Convex Analysis and Optimization

Chapter 6 Solutions

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CHAPTER 6: SOLUTION MANUAL

6.1

Show that the dual of the (infeasible) linear program

$$\begin{aligned} &\text{minimize} \quad x_1 - x_2 \\ &\text{subject to} \quad x \in X = \{x \mid x_1 \geq 0,\, x_2 \geq 0\}, \quad x_1 + 1 \leq 0,\ 1 - x_1 - x_2 \leq 0 \end{aligned}$$

is the (infeasible) linear program

$$\begin{aligned} &\text{maximize} \quad \mu_1 + \mu_2 \\ &\text{subject to} \quad \mu_1 \geq 0,\ \mu_2 \geq 0,\ -\mu_1 + \mu_2 - 1 \leq 0,\ \mu_2 + 1 \leq 0. \end{aligned}$$

Solution: We consider the dual function

$$\begin{aligned} q(\mu_1, \mu_2) &= \inf_{x_1 \geq 0,\ x_2 \geq 0} \big\{ x_1 - x_2 + \mu_1(x_1 + 1) + \mu_2(1 - x_1 - x_2) \big\} \\ &= \inf_{x_1 \geq 0,\ x_2 \geq 0} \big\{ x_1(1 + \mu_1 - \mu_2) + x_2(-1 - \mu_2) + \mu_1 + \mu_2 \big\}. \end{aligned}$$

It can be seen that if $-\mu_1 + \mu_2 - 1 \leq 0$ and $\mu_2 + 1 \leq 0$, then the infimum above is attained at $x_1 = 0$ and $x_2 = 0$. In this case, the dual function is given by $q(\mu_1, \mu_2) = \mu_1 + \mu_2$. On the other hand, if $1 + \mu_1 - \mu_2 < 0$ or $-1 - \mu_2 < 0$, then we have $q(\mu_1, \mu_2) = -\infty$. Thus, the dual problem is

$$\begin{aligned} &\text{maximize} \quad \mu_1 + \mu_2 \\ &\text{subject to} \quad \mu_1 \geq 0,\ \mu_2 \geq 0,\ -\mu_1 + \mu_2 - 1 \leq 0,\ \mu_2 + 1 \leq 0. \end{aligned}$$

6.2 (Extended Representation)

Consider problem (P) and assume that the set X is described by equality and inequality constraints as

$$X = \big\{ x \mid h_i(x) = 0,\, i = m+1, \ldots, \overline{m},\, g_j(x) \leq 0,\, j = r+1, \ldots, \overline{r} \big\}.$$

Then the problem can alternatively be described without an abstract set constraint, in terms of all of the constraint functions

$$h_i(x) = 0, \quad i = 1, \ldots, \overline{m}, \qquad g_j(x) \leq 0, \quad j = 1, \ldots, \overline{r}.$$

We call this the *extended representation* of (P). Show if there exists a geometric multiplier for the extended representation, there exists a geometric multiplier for the original problem (P).

Solution: Assume that there exists a geometric multiplier in the extended representation. This implies that there exist nonnegative scalars $\lambda_1^*, \ldots, \lambda_m^*, \lambda_{m+1}^*, \ldots, \lambda_{\overline{m}}^*$ and $\mu_1^*, \ldots, \mu_r^*, \mu_{r+1}^*, \ldots, \mu_{\overline{r}}^*$ such that

$$f^* = \inf_{x \in \Re^n} \left\{ f(x) + \sum_{i=1}^{\overline{m}} \lambda_i^* h_i(x) + \sum_{j=1}^{\overline{r}} \mu_j^* g_j(x) \right\},$$

implying that

$$f^* \le f(x) + \sum_{i=1}^{\overline{m}} \lambda_i^* h_i(x) + \sum_{j=1}^{\overline{r}} \mu_j^* g_j(x), \qquad \forall\, x \in \Re^n.$$

For any $x \in X$, we have $h_i(x) = 0$ for all $i = m+1, \ldots, \overline{m}$, and $g_j(x) \le 0$ for all $j = r+1, \ldots, \overline{r}$, so that $\mu_j^* g_j(x) \le 0$ for all $j = r+1, \ldots, \overline{r}$. Therefore, it follows from the preceding relation that

$$f^* \le f(x) + \sum_{i=1}^{m} \lambda_i^* h_i(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall\, x \in X.$$

Taking the infimum over all $x \in X$, it follows that

$$\begin{aligned} f^* &\le \inf_{x \in X} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i^* h_i(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\ &\le \inf_{\substack{x \in X,\ h_i(x)=0,\ i=1,\ldots,m \\ g_j(x) \le 0,\ j=1,\ldots,r}} \left\{ f(x) + \sum_{i=1}^{m} \lambda_i^* h_i(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \\ &\le \inf_{\substack{x \in X,\ h_i(x)=0,\ i=1,\ldots,m \\ g_j(x) \le 0,\ j=1,\ldots,r}} f(x) \\ &= f^*. \end{aligned}$$

Hence, equality holds throughout above, showing that the scalars $\lambda_1^*, \ldots, \lambda_m^*$, $\mu_1^*, \ldots, \mu_r^*$ constitute a geometric multiplier for the original representation.

6.3 (Quadratic Programming Duality)

This exercise is an extension of Prop. 6.3.1. Consider the quadratic program

$$\begin{aligned} &\text{minimize} \quad c'x + \tfrac{1}{2} x'Qx \\ &\text{subject to} \quad x \in X, \qquad a_j'x \le b_j, \quad j = 1, \ldots, r, \end{aligned}$$

where X is a polyhedral set, Q is a symmetric positive semidefinite $n \times n$ matrix, $c, a_1, \ldots, a_r$ are vectors in $\Re^n$, and $b_1, \ldots, b_r$ are scalars, and assume that its optimal value is finite. Then there exist at least one optimal solution and at least one geometric multiplier. *Hint*: Use the extended representation of Exercise 6.2.

Solution: Consider the extended representation of the problem in which the linear inequalities that represent the polyhedral part are lumped with the remaining linear inequality constraints. From Prop. 6.3.1, finiteness of the optimal value implies that there exists an optimal solution and a geometric multiplier. From Exercise 6.2, it follows that there exists a geometric multiplier for the original representation of the problem.

6.4 (Sensitivity)

Consider the class of problems

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \qquad g_j(x) \le u_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $u = (u_1, \ldots, u_r)$ is a vector parameterizing the right-hand side of the constraints. Given two distinct values $\bar{u}$ and $\tilde{u}$ of u, let $\bar{f}$ and $\tilde{f}$ be the corresponding optimal values, and assume that $-\infty < \bar{f} < \infty$ and $-\infty < \tilde{f} < \infty$, and that $\bar{\mu}$ and $\tilde{\mu}$ are corresponding geometric multipliers. Show that

$$\tilde{\mu}'(\tilde{u} - \bar{u}) \le \bar{f} - \tilde{f} \le \overline{\mu}'(\tilde{u} - \bar{u}).$$

Solution: We have

$$\overline{f} = \inf_{x \in X} \big\{ f(x) + \overline{\mu}' \big(g(x) - \overline{u} \big) \big\},$$

$$\tilde{f} = \inf_{x \in X} \big\{ f(x) + \tilde{\mu}' \big(g(x) - \tilde{u} \big) \big\}.$$

Let $\overline{q}(\mu)$ denote the dual function of the problem corresponding to $\overline{u}$:

$$\overline{q}(\mu) = \inf_{x \in X} \big\{ f(x) + \mu' \big(g(x) - \overline{u} \big) \big\}.$$

We have

$$\begin{aligned} \overline{f} - \tilde{f} &= \inf_{x \in X} \big\{ f(x) + \overline{\mu}' \big(g(x) - \overline{u} \big) \big\} - \inf_{x \in X} \big\{ f(x) + \tilde{\mu}' \big(g(x) - \tilde{u} \big) \big\} \\ &= \inf_{x \in X} \big\{ f(x) + \overline{\mu}' \big(g(x) - \overline{u} \big) \big\} - \inf_{x \in X} \big\{ f(x) + \tilde{\mu}' \big(g(x) - \overline{u} \big) \big\} + \tilde{\mu}'(\tilde{u} - \overline{u}) \\ &= \overline{q}(\overline{\mu}) - \overline{q}(\tilde{\mu}) + \tilde{\mu}'(\tilde{u} - \overline{u}) \\ &\ge \tilde{\mu}'(\tilde{u} - \overline{u}), \end{aligned}$$

where the last inequality holds because $\overline{\mu}$ maximizes $\overline{q}$.

This proves the left-hand side of the desired inequality. Interchanging the roles of $\overline{f}$, $\overline{u}$, $\overline{\mu}$, and $\tilde{f}$, $\tilde{u}$, $\tilde{\mu}$, shows the desired right-hand side.

6.5

Verify the linear programming duality relations

$$\min_{A'x \geq b} c'x \qquad \Longleftrightarrow \qquad \max_{A\mu = c,\ \mu \geq 0} b'\mu,$$

$$\min_{A'x \geq b,\ x \geq 0} c'x \qquad \Longleftrightarrow \qquad \max_{A\mu \leq c,\ \mu \geq 0} b'\mu,$$

show that they are symmetric, and derive the corresponding complementary slackness conditions [cf. Eqs. (6.13) and (6.14)].

Solution: We first consider the relation

$$(P) \qquad \min_{A'x \geq b} c'x \Longleftrightarrow \max_{A\mu = c, \mu \geq 0} b'\mu. \quad (D)$$

The dual problem to (P) is

$$\max_{\mu \geq 0} q(\mu) = \max_{\mu \geq 0} \inf_{x \in \Re^n} \left\{ \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} \mu_i a_{ij} \right) x_j + \sum_{i=1}^{m} \mu_i b_i \right\}.$$

If $c_j - \sum_{i=1}^{m} \mu_i a_{ij} \neq 0$, then $q(\mu) = -\infty$. Thus the dual problem is

$$\begin{aligned} &\text{maximize} \quad \sum_{i=1}^{m} \mu_i b_i \\ &\text{subject to} \quad \sum_{i=1}^{m} \mu_i a_{ij} = c_j, \quad j = 1, \ldots, n, \quad \mu \geq 0. \end{aligned}$$

To determine the dual of (D), note that (D) is equivalent to

$$\min_{A\mu = c, \mu \geq 0} -b'\mu,$$

and so its dual problem is

$$\max_{x \in \Re^n} p(x) = \max_{x} \inf_{\mu \geq 0} \big\{ (Ax - b)'\mu - c'x \big\}.$$

If $a_i'x - b_i < 0$ for any i, then $p(x) = -\infty$. Thus the dual of (D) is

$$\begin{aligned} &\text{maximize} \quad -c'x \\ &\text{subject to} \quad A'x \geq b, \end{aligned}$$

or

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad A'x \geq b. \end{aligned}$$

The Lagrangian optimality condition for (P) is

$$x^* = \arg\min_x \left\{ \left(c - \sum_{i=1}^m \mu_i^* a_i \right)' x + \sum_{i=1}^m \mu_i^* b_i \right\},$$

from which we obtain the complementary slackness conditions for (P):

$$A\mu = c.$$

The Lagrangian optimality condition for (D) is

$$\mu^* = \arg\min_{\mu \geq 0} \{ (Ax^* - b)'\mu - c'x^* \},$$

from which we obtain the complementary slackness conditions for (D):

$$Ax^* - b \geq 0, \qquad (Ax^* - b)_i \mu_i^* = 0, \quad \forall\, i.$$

Next, consider

$$(P) \qquad \min_{A'x \geq b, x \geq 0} c'x \Longleftrightarrow \max_{A\mu \leq c, \mu \geq 0} b'\mu. \quad (D)$$

The dual problem to (P) is

$$\max_{\mu \geq 0} q(\mu) = \max_{\mu \geq 0} \inf_{x \geq 0} \left\{ \sum_{j=1}^n \left(c_j - \sum_{i=1}^m \mu_i a_{ij} \right) x_j + \sum_{i=1}^m \mu_i b_i \right\}.$$

If $c_j - \sum_{i=1}^m \mu_i a_{ij} < 0$, then $q(\mu) = -\infty$. Thus the dual problem is

$$\begin{aligned} &\text{maximize} \quad \sum_{i=1}^m \mu_i b_i \\ &\text{subject to} \quad \sum_{i=1}^m \mu_i a_{ij} \leq c_j, \quad j = 1, \ldots, n, \quad \mu \geq 0. \end{aligned}$$

To determine the dual of (D), note that (D) is equivalent to

$$\min_{A\mu \leq c, \mu \geq 0} -b'\mu,$$

and so its dual problem is

$$\max_{x \geq 0} p(x) = \max_{x \geq 0} \inf_{\mu \geq 0} \big\{ (Ax - b)'\mu - c'x \big\}.$$

If $a_i'x - b_i < 0$ for any i, then $p(x) = -\infty$. Thus the dual of (D) is

$$\begin{aligned} &\text{maximize} \quad -c'x \\ &\text{subject to} \quad A'x \geq b, \quad x \geq 0 \end{aligned}$$

or

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad A'x \ge b, \quad x \ge 0. \end{aligned}$$

The Lagrangian optimality condition for (P) is

$$x^* = \arg\min_{x \ge 0} \left\{ \left(c - \sum_{i=1}^m \mu_i^* a_i \right)' x + \sum_{i=1}^m \mu_i^* b_i \right\},$$

from which we obtain the complementary slackness conditions for (P):

$$\left(c_j - \sum_{i=1}^m \mu_i^* a_{ij} \right) x_j^* = 0, \quad x_j^* \ge 0, \quad \forall\, j = 1, \ldots, n,$$

$$c - \sum_{i=1}^m \mu_i^* a_i \ge 0.$$

The Lagrangian optimality condition for (D) is

$$\mu^* = \arg\min_{\mu \ge 0} \big\{ (Ax^* - b)'\mu - c'x^* \big\},$$

from which we obtain the complementary slackness conditions for (D):

$$Ax^* - b \ge 0, \qquad (Ax^* - b)_i \mu_i^* = 0, \quad \forall\, i.$$

6.6 (Duality and Zero Sum Games)

Let A be an $n \times m$ matrix, and let X and Z be the unit simplices in $\Re^n$ and $\Re^m$, respectively:

$$X = \left\{ x \;\middle|\; \sum_{i=1}^n x_i = 1,\, x_i \ge 0,\, i = 1, \ldots, n \right\},$$

$$Z = \left\{ z \;\middle|\; \sum_{j=1}^m z_j = 1,\, z_j \ge 0,\, j = 1, \ldots, m \right\}.$$

Show that the minimax equality

$$\max_{z \in Z} \min_{x \in X} x'Az = \min_{x \in X} \max_{z \in Z} x'Az$$

is a special case of linear programming duality. *Hint*: For a fixed z, $\min_{x \in X} x'Az$ is equal to the minimum component of the vector Az, so

$$\max_{z \in Z} \min_{x \in X} x'Az = \max_{z \in Z} \min\big\{ (Az)_1, \ldots, (Az)_n \big\} = \max_{\xi e \le Az,\, z \in Z} \xi, \tag{6.76}$$

where e is the unit vector in $\Re^n$ (all components are equal to 1). Similarly,

$$\min_{x\in X}\max_{z\in Z} x'Az = \min_{\zeta e\geq A'x,\, x\in X} \zeta. \tag{6.77}$$

Show that the linear programs in the right-hand sides of Eqs. (6.76) and (6.77) are dual to each other.

Solution: Consider the linear program

$$\min_{\substack{\zeta e\geq A'x \\ \sum_{i=1}^n x_i=1,\, x_i\geq 0}} \zeta,$$

whose optimal value is equal to $\min_{x\in X}\max_{z\in Z} x'Az$. Introduce dual variables $z\in\Re^m$ and $\xi\in\Re$, corresponding to the constraints $A'x-\zeta e\leq 0$ and $\sum_{i=1}^n x_i = 1$, respectively. The dual function is

$$\begin{aligned} q(z,\xi) &= \inf_{x_i\geq 0,\, i=1,\ldots,n} \left\{ \zeta + z'(A'x-\zeta e) + \xi\left(1-\sum_{i=1}^n x_i\right)\right\} \\ &= \inf_{x_i\geq 0,\, i=1,\ldots,n} \left\{ \zeta\left(1-\sum_{j=1}^m z_j\right) + x'(Az-\xi e) + \xi\right\} \\ &= \begin{cases} \xi & \text{if } \sum_{j=1}^m z_j = 1,\ \xi e - Az\leq 0, \\ -\infty & \text{otherwise.}\end{cases} \end{aligned}$$

Thus the dual problem, which is to maximize $q(z,\xi)$ subject to $z\geq 0$ and $\xi\in\Re$, is equivalent to the linear program

$$\max_{\xi e\leq Az,\, z\in Z} \xi,$$

whose optimal value is equal to $\max_{z\in Z}\min_{x\in X} x'Az$.

6.7 (Goldman-Tucker Complementarity Theorem [GoT56])

Consider the linear programming problem

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad Ax = b, \qquad x\geq 0, \end{aligned} \tag{LP}$$

where A is an $m\times n$ matrix, c is a vector in $\Re^n$, and b is a vector in $\Re^m$. Consider also the dual problem

$$\begin{aligned} &\text{maximize} \quad b'\lambda \\ &\text{subject to} \quad A'\lambda\leq c. \end{aligned} \tag{DLP}$$

Assume that the sets of optimal solutions of LP and DLP, denoted X^* and Λ^*, respectively, are nonempty. Show that the index set $\{1,\ldots,n\}$ can be partitioned into two disjoint subsets I and $\overline{I}$ with the following two properties:

(1) For all $x^* \in X^*$ and $\lambda^* \in \Lambda^*$, we have

$$x_i^* = 0, \qquad \forall\, i \in \overline{I}, \qquad (A'\lambda^*)_i = c_i, \qquad \forall\, i \in I,$$

where x_i^* and $(A'\lambda^*)_i$ are the ith components of x^* and $A'\lambda^*$, respectively.

(2) There exist vectors $x^* \in X^*$ and $\lambda^* \in \Lambda^*$ such that

$$x_i^* > 0, \quad \forall\, i \in I, \qquad x_i^* = 0, \quad \forall\, i \in \overline{I},$$

$$(A'\lambda^*)_i = c_i, \quad \forall\, i \in I, \qquad (A'\lambda^*)_i < c_i, \quad \forall\, i \in \overline{I}.$$

Hint: Apply the Tucker Complementarity Theorem (Exercise 3.32).

Solution: Consider the subspace

$$S = \big\{(x, w) \mid bw - Ax = 0,\ c'x = wv,\ x \in \Re^n,\ w \in \Re\big\},$$

where v is the optimal value of (LP). Its orthogonal complement is the range of the matrix

$$\begin{bmatrix} -A' & c \\ b & -v \end{bmatrix},$$

so it has the form

$$S^\perp = \big\{(c\zeta - A'\lambda, b'\lambda - v\zeta) \mid \lambda \in \Re^m,\ \zeta \in \Re\big\}.$$

Applying the Tucker Complementarity Theorem (Exercise 3.32) for this choice of S, we obtain a partition of the index set $\{1, \ldots, n+1\}$ in two subsets. There are two possible cases: (1) the index $n+1$ belongs to the first subset, or (2) the index $n+1$ belongs to the second subset. Since the vectors $(x, 1)$ such that $x \in X^*$ satisfy $Ax - bw = 0$ and $c'x = wv$, we see that case (1) holds, i.e., the index $n+1$ belongs to the first index subset. In particular, we have that there exist disjoint index sets I and $\overline{I}$ such that $I \cup \overline{I} = \{1, \ldots, n\}$ and the following properties hold:

(a) There exist vectors $(x, w) \in S$ and $(\lambda, \zeta) \in \Re^{m+1}$ with the property

$$x_i > 0, \quad \forall\, i \in I, \qquad x_i = 0, \quad \forall\, i \in \overline{I}, \qquad w > 0, \tag{6.1}$$

$$c_i\zeta - (A'\lambda)_i = 0, \quad \forall\, i \in I, \quad c_i\zeta - (A'\lambda)_i > 0, \quad \forall\, i \in \overline{I}, \qquad b'\lambda = v\zeta. \tag{6.2}$$

(b) For all $(x, w) \in S$ with $x \geq 0$, and $(\lambda, \zeta) \in \Re^{m+1}$ with $c\zeta - A'\lambda \geq 0$, $v\zeta - b'\lambda \geq 0$, we have

$$x_i = 0, \qquad \forall\, i \in \overline{I},$$

$$c_i\zeta - (A'\lambda)_i = 0, \qquad \forall\, i \in I, \qquad b'\lambda = v\zeta.$$

By dividing (x, w) by w, we obtain [cf. Eq. (6.1)] an optimal primal solution $x^* = x/w$ such that

$$x_i^* > 0, \quad \forall\, i \in I, \qquad x_i^* = 0, \quad \forall\, i \in \overline{I}.$$

Similarly, if the scalar ζ in Eq. (6.2) is positive, by dividing with ζ in Eq. (6.2), we obtain an optimal dual solution $\lambda^* = \lambda/\zeta$, which satisfies the desired property

$$c_i - (A'\lambda^*)_i = 0, \quad \forall\, i \in I, \qquad c_i - (A'\lambda^*)_i > 0, \quad \forall\, i \in \overline{I}.$$

If the scalar ζ in Eq. (6.2) is nonpositive, we choose any optimal dual solution λ^*, and we note, using also property (b), that we have

$$c_i-(A'\lambda^*)_i = 0, \qquad \forall\, i \in I, \qquad c_i-(A'\lambda^*)_i \geq 0, \quad \forall\, i \in \overline{I}, \qquad b'\lambda^* = v. \quad (6.3)$$

Consider the vector

$$\tilde{\lambda} = (1-\zeta)\lambda^* + \lambda.$$

By multiplying Eq. (6.3) with the positive number $1 - \zeta$, and by combining it with Eq. (6.2), we see that

$$c_i - (A'\tilde{\lambda})_i = 0, \qquad \forall\, i \in I, \qquad c_i - (A'\tilde{\lambda})_i > 0, \quad \forall\, i \in \overline{I}, \qquad b'\tilde{\lambda} = v.$$

Thus, $\tilde{\lambda}$ is an optimal dual solution that satisfies the desired property.

6.8

Use duality to show that in three-dimensional space, the (minimum) distance from the origin to a line is equal to the maximum over all (minimum) distances of the origin from planes that contain the line.

Solution: The problem of finding the minimum distance from the origin to a line is written as

$$\min \tfrac{1}{2}\|x\|^2$$

$$\text{subject to } Ax = b,$$

where A is a 2×3 matrix with full rank, and $b \in \Re^2$. Let f^* be the optimal value and consider the dual function

$$q(\lambda) = \min_x \left\{ \tfrac{1}{2}\|x\|^2 + \lambda'(Ax-b) \right\}.$$

By Prop. 6.3.1, since the optimal value is finite, it follows that this problem has no duality gap.

Let V^* be the supremum over all distances of the origin from planes that contain the line $\{x \mid Ax = b\}$. Clearly, we have $V^* \leq f^*$, since the distance to the line $\{x \mid Ax = b\}$ cannot be smaller than the distance to the plane that contains the line.

We now note that any plane of the form $\{x \mid p'Ax = p'b\}$, where $p \in \Re^2$, contains the line $\{x \mid Ax = b\}$, so we have for all $p \in \Re^2$,

$$V(p) \equiv \min_{p'Ax=p'x} \tfrac{1}{2}\|x\|^2 \leq V^*.$$

On the other hand, by duality in the minimization of the preceding equation, we have

$$U(p,\gamma) \equiv \min_x \left\{ \tfrac{1}{2}\|x\|^2 + \gamma(p'Ax - p'x) \right\} \le V(p), \qquad \forall\, p \in \Re^2,\, \gamma \in \Re.$$

Combining the preceding relations, it follows that

$$\sup_\lambda q(\lambda) = \sup_{p,\gamma} U(p,\gamma) \le \sup_p U(p,1) \le \sup_p V(p) \le V^* \le f^*.$$

Since there is no duality gap for the original problem, we have $\sup_\lambda q(\lambda) = f^*$, it follows that equality holds throughout above. Hence $V^* = f^*$, which was to be proved.

6.9

Consider the problem

$$\begin{aligned} &\text{minimize} \quad \sum_{i=0}^m f_i(x) \\ &\text{subject to} \quad x \in X_i, \qquad i = 0,1,\ldots,m, \end{aligned}$$

where $f_i : \Re^n \mapsto \Re$ are convex functions and X_i are bounded polyhedral subsets of $\Re^n$ with nonempty intersection. Show that a dual problem is given by

$$\begin{aligned} &\text{maximize} \quad q_0(\lambda_1 + \cdots + \lambda_m) + \sum_{i=1}^m q_i(\lambda_i) \\ &\text{subject to} \quad \lambda_i \in \Re^n, \qquad i = 1,\ldots,m, \end{aligned}$$

where the functions $q_i : \Re^n \mapsto \Re$ are given by

$$q_0(\lambda) = \min_{x \in X_0} \left\{ f_0(x) - \lambda' x \right\},$$

$$q_i(\lambda) = \min_{x \in X_i} \left\{ f_i(x) + \lambda' x \right\}, \qquad i = 1,\ldots,m.$$

Show also that the primal and dual problems have optimal solutions, and that there is no duality gap. *Hint*: Introduce artificial optimization variables $z_1,\ldots,z_m$ and the linear constraints $x = z_i$, $i = 1,\ldots,m$.

Solution: We introduce artificial variables $x_0, x_1, \ldots, x_m$, and we write the problem in the equivalent form

$$\begin{aligned} &\text{minimize} \quad \sum_{i=0}^m f_i(x_i) \\ &\text{subject to}\ x_i \in X_i, \quad i = 0,\ldots,m \qquad x_i = x_0, \quad i = 1,\ldots,m. \end{aligned} \tag{6.4}$$

By relaxing the equality constraints, we obtain the dual function

$$q(\lambda_1, \ldots, \lambda_m) = \inf_{x_i \in X_i,\ i=0,\ldots,m} \left\{ \sum_{i=0}^{m} f_i(x_i) + \lambda_i'(x_i - x_0) \right\}$$

$$= \inf_{x \in X_0} \big\{ f_0(x) - (\lambda_1 + \cdots \lambda_m)'x \big\} + \sum_{i=1}^{m} \inf_{x \in X_i} \big\{ f_i(x) + \lambda_i' x \big\},$$

which is of the form given in the exercise. Note that the infima above are attained since f_i are continuous (being convex functions over $\Re^n$) and X_i are compact polyhedra.

Because the primal problem involves minimization of the continuous function $\sum_{i=0}^{m} f_i(x)$ over the compact set $\cap_{i=0}^{m} X_i$, a primal optimal solution exists. Applying Prop. 6.4.2 to problem (6.4), we see that there is no duality gap and there exists at least one geometric multiplier, which is a dual optimal solution.

6.10

Consider the problem

$$\text{minimize} \quad f(x)$$
$$\text{subject to} \quad x \in X, \quad g_j(x) \leq 0, \ j = 1, \ldots, r,$$

and assume that f^* is finite, X is convex, and the functions $f : \Re^n \mapsto \Re$ and $g_j : \Re^n \mapsto \Re$ are convex over X. Show that if the set of geometric multipliers is nonempty and compact, then the Slater condition holds.

Solution: Let M denote the set of geometric multipliers, i.e.,

$$M = \left\{ \mu \geq 0 \;\Big|\; f^* = \inf_{x \in X} \big\{ f(x) + \mu' g(x) \big\} \right\}.$$

We will show that if the set M is nonempty and compact, then the Slater condition holds. Indeed, if this were not so, then 0 would not be an interior point of the set

$$D = \big\{ u \mid \text{there exists some } x \in X \text{ such that } g(x) \leq u \big\}.$$

By a similar argument as in the proof of Prop. 6.6.1, it can be seen that D is convex. Therefore, we can use the Supporting Hyperplane Theorem to assert the existence of a hyperplane that passes through 0 and contains D in its positive halfspace, i.e., there is a nonzero vector $\overline{\mu}$ such that $\overline{\mu}'u \geq 0$ for all $u \in D$. This implies that $\overline{\mu} \geq 0$, since for each $u \in D$, we have that $(u_1, \ldots, u_j + \gamma, \ldots, u_r) \in D$ for all $\gamma > 0$ and j. Since $g(x) \in D$ for all $x \in X$, it follows that

$$\overline{\mu}' g(x) \geq 0, \qquad \forall\ x \in X.$$

Thus, for any $\mu \in M$, we have

$$f(x) + (\mu + \gamma \overline{\mu})' g(x) \geq f^*, \qquad \forall\ x \in X,\ \forall\ \gamma \geq 0.$$

Hence, it follows that $(\mu + \gamma \overline{\mu}) \in M$ for all $\gamma \geq 0$, which contradicts the boundedness of M.

6.11 (Inconsistent Convex Systems of Inequalities)

Let $g_j : \Re^n \mapsto \Re$, $j = 1, \ldots, r$, be convex functions over the nonempty convex subset of $\Re^n$. Show that the system

$$g_j(x) < 0, \qquad j = 1, \ldots, r,$$

has no solution within X if and only if there exists a vector $\mu \in \Re^r$ such that

$$\sum_{j=1}^{r} \mu_j = 1, \qquad \mu \geq 0,$$

$$\mu' g(x) \geq 0, \qquad \forall\, x \in X.$$

Hint: Consider the convex program

$$\begin{aligned} &\text{minimize } \ y \\ &\text{subject to } \ x \in X, \quad y \in \Re, \qquad g_j(x) \leq y, \quad j = 1, \ldots, r. \end{aligned}$$

Solution: The dual function for the problem in the hint is

$$\begin{aligned} q(\mu) &= \inf_{y \in \Re,\, x \in X} \left\{ y + \sum_{j=1}^{r} \mu_j \big(g_j(x) - y \big) \right\} \\ &= \begin{cases} \inf_{x \in X} \sum_{j=1}^{r} \mu_j g_j(x) & \text{if } \sum_{j=1}^{r} \mu_j = 1, \\ -\infty & \text{if } \sum_{j=1}^{r} \mu_j \neq 1. \end{cases} \end{aligned}$$

The problem in the hint satisfies Assumption 6.4.2, so by Prop. 6.4.3, the dual problem has an optimal solution μ^* and there is no duality gap.

Clearly the problem in the hint has an optimal value that is greater or equal to 0 if and only if the system of inequalities

$$g_j(x) < 0, \qquad j = 1, \ldots, r,$$

has no solution within X. Since there is no duality gap, we have

$$\max_{\mu \geq 0,\, \sum_{j=1}^{r} \mu_j = 1} q(\mu) \geq 0$$

if and only if the system of inequalities $g_j(x) < 0$, $j = 1, \ldots, r$, has no solution within X. This is equivalent to the statement we want to prove.

6.12

This exercise is a refinement of the Enhanced Farkas' Lemma (Prop. 5.4.2). Let N be a closed cone in $\Re^n$, let $a_1, \ldots, a_r$ be vectors in $\Re^n$, and let c be a vector in $\text{cone}\big(\{a_1, \ldots, a_r\}\big) + \text{ri}(N)$ such that $c \notin N$. Show that there is a nonempty index set $J \subset \{1, \ldots, r\}$ such that:

(1) The vector c can be represented as a positive combination of the vectors a_j, $j \in J$, plus a vector in N.

(2) There is a hyperplane that passes through the origin, and contains the vectors a_j, $j \in J$, in one of its open halfspaces and the vectors a_j, $j \notin J$, in the complementary closed halfspace.

Hint: Combine Example 6.4.2 with Lemma 5.3.1.

Solution: Since $c \in \text{cone}\{a_1, \ldots, a_r\} + \text{ri}(N)$, there exists a vector $\overline{\mu} \geq 0$ such that

$$-\left(-c + \sum_{j=1}^{r} \overline{\mu}_j a_j\right) \in \text{ri}(N).$$

By Example 6.4.2, this implies that the problem

$$\begin{aligned} \text{minimize} \quad & -c'd + \tfrac{1}{2} \textstyle\sum_{j=1}^{r} \big((a_j'd)^+\big)^2 \\ \text{subject to} \quad & d \in N^*, \end{aligned}$$

has an optimal solution, which we denote by d^*. Consider the set

$$M = \left\{ \mu \geq 0 \;\middle|\; -\left(-c + \sum_{j=1}^{r} \mu_j a_j\right) \in N \right\},$$

which is nonempty by assumption. Let μ^* be the vector of minimum norm in M and let the index set J be defined by

$$J = \{ j \mid \mu_j^* > 0 \}.$$

Then, it follows from Lemma 5.3.1 that

$$a_j' d^* > 0, \qquad \forall\, j \in J,$$

and

$$a_j' d^* \leq 0, \qquad \forall\, j \notin J,$$

thus proving that the properties (1) and (2) of the exercise hold.

6.13 (Pareto Optimality)

A decisionmaker wishes to choose a vector $x \in X$, which keeps the values of *two* cost functions $f_1 : \Re^n \mapsto \Re$ and $f_2 : \Re^n \mapsto \Re$ reasonably small. Since a vector x^* minimizing simultaneously both f_1 and f_2 over X need not exist, he/she decides to settle for a *Pareto optimal solution*, i.e., a vector $x^* \in X$ with the property that there does not exist any vector $\bar{x} \in X$ that is strictly better than x^*, in the sense that either

$$f_1(\bar{x}) \le f_1(x^*), \qquad f_2(\bar{x}) < f_2(x^*),$$

or

$$f_1(\bar{x}) < f_1(x^*), \qquad f_2(\bar{x}) \le f_2(x^*).$$

(a) Show that if x^* is a vector in X, and λ_1^* and λ_2^* are two positive scalars such that

$$\lambda_1^* f_1(x^*) + \lambda_2^* f_2(x^*) = \min_{x \in X} \big\{ \lambda_1^* f_1(x) + \lambda_2^* f_2(x) \big\},$$

then x^* is a Pareto optimal solution.

(b) Assume that X is convex and f_1, f_2 are convex over X. Show that if x^* is a Pareto optimal solution, then there exist non-negative scalars λ_1^*, λ_2^*, not both zero, such that

$$\lambda_1^* f_1(x^*) + \lambda_2^* f_2(x^*) = \min_{x \in X} \big\{ \lambda_1^* f_1(x) + \lambda_2^* f_2(x) \big\}.$$

Hint: Consider the set

$$A = \Big\{ (z_1, z_2) \mid \text{there exists } x \in X \text{ such that } f_1(x) \le z_1,\ f_2(x) \le z_2 \Big\}$$

and show that it is a convex set. Use hyperplane separation arguments.

(c) Generalize the results of (a) and (b) to the case where there are m cost functions rather than two.

Solution: (a) Assume that x^* is not a Pareto optimal solution. Then there is a vector $\overline{x} \in X$ such that either

$$f_1(\bar{x}) \le f_1(x^*), \quad f_2(\bar{x}) < f_2(x^*),$$

or

$$f_1(\bar{x}) < f_1(x^*), \quad f_2(\bar{x}) \le f_2(x^*).$$

Multiplying the left equation by λ_1^*, the right equation by λ_2^*, and adding the two in either case yields

$$\lambda_1^* f_1(\bar{x}) + \lambda_2^* f_2(\bar{x}) < \lambda_1^* f_1(x^*) + \lambda_2^* f_2(x^*),$$

yielding a contradiction. Therefore x^* is a Pareto optimal solution.

(b) Let

$$A = \{(z_1, z_2) \mid \text{there exists } x \in X \text{ such that } f_1(x) \le z_1, f_2(x) \le z_2\}.$$

We first show that A is convex. Indeed, let (a_1, a_2), and (b_1, b_2) be elements of A, and let $(c_1, c_2) = \alpha(a_1, a_2) + (1-\alpha)(b_1, b_2)$ for any $\alpha \in [0, 1]$. Then for some $x_a \in X$, $x_b \in X$, we have $f_1(x_a) \le a_1$, $f_2(x_a) \le a_2$, $f_1(x_b) \le b_1$, and $f_2(x_b) \le b_2$. Let $x_c = \alpha x_a + (1-\alpha)x_b$. Since X is convex, $x_c \in X$. Since f is convex, we also have

$$f_1(x_c) \le c_1, \quad \text{and} \quad f_2(x_c) \le c_2.$$

Hence, $(c_1, c_2) \in A$ and it follows that A is a convex set.

For any $x \in X$, we have $\big(f_1(x), f_2(x)\big) \in A$. In addition, $\big(f_1(x^*), f_2(x^*)\big)$ is in the boundary of A. [If this were not the case, then either (1) or (2) would hold and x^* would not be Pareto optimal.] Then by the Supporting Hyperplane Theorem, there exists λ_1^* and λ_2^*, not both equal to 0, such that

$$\lambda_1^* z_1 + \lambda_2^* z_2 \ge \lambda_1^* f_1(x^*) + \lambda_2^* f_2(x^*), \qquad \forall\ (z_1, z_2) \in A.$$

Since z_1 and z_2 can be made arbitrarily large, we must have $\lambda_1^*, \lambda_2^* \ge 0$. Since $\big(f_1(x), f_2(x)\big) \in A$, the above equation yields

$$\lambda_1^* f_1(x) + \lambda_2^* f_2(x) \ge \lambda_1^* f_1(x^*) + \lambda_2^* f_2(x^*), \quad \forall x \in X,$$

or, equivalently,

$$\min_{x \in X} \big\{\lambda_1^* f_1(x) + \lambda_2^* f_2(x)\big\} \ge \lambda_1^* f_1(x^*) + \lambda_2^* f_2(x^*).$$

Combining this with the fact that

$$\min_{x \in X} \big\{\lambda_1^* f_1(x) + \lambda_2^* f_2(x)\big\} \le \lambda_1^* f_1(x^*) + \lambda_2^* f_2(x^*)$$

yields the desired result.

(c) Generalization of (a): If x^* is a vector in X, and $\lambda_1^*, \ldots, \lambda_m^*$ are positive scalars such that

$$\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = \min_{x \in X} \left\{ \sum_{i=1}^{m} \lambda_i^* f_i(x) \right\},$$

then x^* is a Pareto optimal solution.

Generalization of (b): Assume that X is convex and $f_1, \ldots, f_m$ are convex over X. If x^* is a Pareto optimal solution, then there exist non-negative scalars $\lambda_1^*, \ldots, \lambda_m^*$, not all zero, such that

$$\sum_{i=1}^{m} \lambda_i^* f_i(x^*) = \min_{x \in X} \left\{ \sum_{i=1}^{m} \lambda_i^* f_i(x) \right\}.$$

6.14 (Polyhedral Programming)

Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X, \qquad g_j(x) \le 0, \quad j = 1, \ldots, r, \end{array}$$

where X is a polyhedral set, and f and g_j are real-valued polyhedral functions. Assume that the optimal value is finite. Show that the primal function is proper and polyhedral.

Solution: Using Prop. 3.2.3, it follows that

$$g_j(x) = \max_{i=1,\ldots,m} \big\{ a_{ij}'x + b_{ij} \big\},$$

where a_{ij} are vectors in $\Re^n$ and b_{ij} are scalars. Hence the constraint functions can equivalently be represented as

$$a_{ij}'x + b_{ij} \le 0, \qquad \forall\ i = 1, \ldots, m,\ \forall\ j = 1, \ldots, r.$$

By assumption, the set X is a polyhedral set, and the cost function f is a polyhedral function, hence convex over $\Re^n$. Therefore, we can use the Strong Duality Theorem for linear constraints (cf. Prop. 6.4.2) to conclude that there is no duality gap and there exists at least one geometric multiplier, i.e., there exists a nonnegative vector μ such that

$$f^* = \inf_{x \in X} \big\{ f(x) + \mu' g(x) \big\}.$$

Let $p(u)$ denote the primal function for this problem. The preceding relation implies that

$$\begin{aligned} p(0) - \mu' u &= \inf_{x \in X} \Big\{ f(x) + \mu' \big(g(x) - u \big) \Big\} \\ &\le \inf_{x \in X,\ g(x) \le u} \Big\{ f(x) + \mu' \big(g(x) - u \big) \Big\} \\ &\le \inf_{x \in X,\ g(x) \le u} f(x) \\ &= p(u), \end{aligned}$$

which, in view of the assumption that $p(0)$ is finite, shows that $p(u) > -\infty$ for all $u \in \Re^r$.

The primal function can be obtained by partial minimization as

$$p(u) = \inf_{x \in \Re^n} F(x, u),$$

where

$$F(x, u) = \begin{cases} f(x) & \text{if } g_j(x) \le u_j\ \forall\ j,\ x \in X, \\ \infty & \text{otherwise.} \end{cases}$$

Since, by assumption f is polyhedral, the g_j are polyhedral (which implies that the level sets of the g_j are polyhedral), and X is polyhedral, it follows that $F(x, u)$ is a polyhedral function. Since we have also shown that $p(u) > -\infty$ for all $u \in \Re^r$, we can use Exercise 3.13 to conclude that the primal function p is polyhedral, and therefore also closed. Since $p(0)$, the optimal value, is assumed finite, it follows that p is proper.