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The non-existence of absolutely continuous invariant probabilities is C^1-generic for flows

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Tese de Doutorado

Thesis presented to the Programa de Pós-Graduação em Matemática of the Departamento de Matemática, PUC–Rio, as partial fulfillment of the requirements for the degree of Doutor em Matemática.

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In memory of my father. His incredible wisdom and irresistible tenderness continue to inspire me.

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Abstract

Koiller Schnoor, Miguel Adriano; Bochi, Jairo. **The non-existence of absolutely continuous invariant probabilities is C^1-generic for flows**. Rio de Janeiro, 2012. 79p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

We prove that C^1-generic vector fields in a compact manifold do not have absolutely continuous invariant probabilities. This extends a result of Avila and Bochi to the continuous time case.

Keywords

Absolutely continuous invariant probability; ergodic theory; non-invariant Rokhlin tower; orthonormal frame flow.

Resumo

Koiller Schnoor, Miguel Adriano; Bochi, Jairo. **Fluxos C^1-genéricos não possuem probabilidades invariantes absolutamente contínuas**. Rio de Janeiro, 2012. 79p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Provamos que campos de vetores C^1-genéricos em uma variedade compacta não possuem probabilidades invariantes absolutamente contínuas em relação a uma medida de volume. Este trabalho estende ao caso de tempo contínuo um resultado de Avila e Bochi.

Palavras–chave

Fluxo de frames ortonormais; probabilidade invariante absolutamente contínua; teoria ergódica; torre de Rokhlin não invariante.

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1
Introduction

Throughout the text, let M be a smooth compact Riemannian manifold without boundary, and let m be the normalized Riemannian volume. The space of all C^r vector fields on M endowed with the C^r topology will be denoted by $\mathfrak{X}^r(M)$. The flow induced by a vector field $X \in \mathfrak{X}^1(M)$ will be denoted by $\{\varphi_X^t\}_{t\in\mathbb{R}}$ or simply $\{\varphi^t\}_{t\in\mathbb{R}}$ if the generating vector field is clear from the context. Let *acip* stand for absolutely continuous invariant probability, where absolute continuity is understood with respect to the volume measure m.

1.1
Absolutely continuous invariant probabilities

The main aspect of invariant probabilities is that they reflect the asymptotical behavior of almost every point with respect to those measures. Although the Krylov-Bogolubov Theorem guarantees the existence of invariant probabilities for compact metrizable spaces, it does not give any other information about the measure. The invariant measure on a Riemannian manifold could be singular to respect to the Riemannian volume. On the other hand, if an invariant probability is absolutely continuous with respect to a volume measure, then it is guaranteed that it reflects the asymptotical behavior of points in a set with positive volume.

The problem of dealing with acips is that, except for the case of $C^{1+\alpha}$ expanding maps (which always admit an acip), it is not known of any other system for which the existence of acips is open (in any topology). Even in the context of expanding maps, it was shown by A. Quas ([Q]) in dimension one and generalized by Avila and Boch in any dimension ([AB1]), that C^1-generic invariant probabilities are singular. Avila and Bochi also generalized Quas result (in the one dimensional case) for σ-finite measures ([AB2]). In the space of C^k Anosov Systems, the absence of acips is an open and dense property. This follows from the Livsic periodic orbit criterion (See [L]).

1.2
Main Theorem

In this work we extend the result of [AB1] for C^1 flows. Let us state precisely the theorem we prove.

Theorem 1.2.1 *There exists a C^1-residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$, then X has no acip.*

Notice that we are not assuming any regularity on the density of the acip, other then integrability. If we ask the acip to be smooth or even holder-continuous, the proof might be much simpler. Our strategy (like in [AB1]) does not need to use these stronger hypotheses.

We assume that M has dimension $d \geq 3$. There is no loss of generality to do so, since the 2-dimensional case is a consequence of the fact that Morse-Smale systems cannot admit an acip (Remark 2.5.3) and the following celebrated result:

Theorem 1.2.2 (Peixoto, Pugh) *Let M be a compact surface. The set of all Morse-Smale systems is (open and) dense in $\mathfrak{X}^1(M)$.*

This theorem was proved for orientable surfaces (and a few non-orientable ones) by M. Peixoto (actually in any C^r topology), and then for every surface (in the C^1 topology) by C. Pugh, using the Closing Lemma. See [PdM, Chapter IV].

Peixoto's original result points to the possibility that the lack of acips might be generic even in higher topologies, since it implies that this is true at least for orientable surfaces.

1.3
Remarks about the proof

The idea of the proof is similar to [AB1]. We consider for each $\delta \in (0,1)$, the set

$$\mathcal{V}_\delta = \big\{ X \in \mathfrak{X}^1(M) : \text{there exist a Borel set } K \subset M \text{ and } T \in \mathbb{R} \text{ such that} \\ m(K) > 1 - \delta \;\text{ and }\; m(\varphi_X^T(K)) < \delta \big\}$$

These sets are clearly open (as shown in Remark 2.5.4); thus if we prove that they are C^1 dense, then the set

$$\mathcal{R} \equiv \bigcap_{\delta \in \mathbb{Q} \cap (0,1)} \mathcal{V}_\delta$$

will be a residual set. The fact that a vector field in $\mathcal{R}$ does not admit an acip is a direct consequence of Lemma 2.5.1. We say that a vector field is δ-*crushing* if $X \in \mathcal{V}_\delta$. All our effort in this work is to prove that δ-crushing is a dense property. Thus we begin with an arbitrary $X \in \mathfrak{X}^1(M)$ and a constant $\delta \in (0,1)$ and show how to construct a perturbation of X with the δ-crushing property.

The strategy to prove denseness of $\mathcal{V}_\delta$ has two main parts. First, we show how to construct a perturbation of X supported on a tubular neighborhood of a very long segment of orbit in a way that the δ-crushing property with respect to the normalized volume can be verified inside this neighborhood. This is the content of the Fettuccine's Lemma (Lemma 5.0.22).

The next step is to show that we can cover the manifold (except for a negligible measure set) with "crushable" sets, permitting us to construct the perturbation globally and, consequently, to obtain the δ-crushing property with respect to the volume of the whole manifold. This is done by a combination of Lemma 3.0.6, where we construct a transverse section and a first return map with some nice properties, and Lemma 6.0.2, which gives us a Rokhlin-like tower with respect to that first return map.

Although the general idea of the proof follows [AB1], there are some difficulties in adapting the proof to the continuous-time case. In both cases, the crushing is done in one dimension only, making d-dimensional objects essentially $(d-1)$-dimensional. In the continuous case, the choice of the crushing direction and the construction of the perturbation is done with the help of a tubular chart with several technical properties (Theorem 4.0.15), while in the discrete setting, an atlas is fixed with the only requirement that charts on the atlas take the volume in M to the Lebesgue measure in $\mathbb{R}^d$.

In [AB1], the crushable sets are contained in a discrete open tower and it is possible, in that case, to make 'a priori' adjustments, like a linearizing perturbation of the map in each level of the tower or a rotation of coordinates that makes $\mathbb{R}^{d-1} \times \{0\}$ invariant by the linear perturbed map. Moreover, these adjustments make the discrete version of Fettuccine's Lemma ([AB1, Lemma 3]) much simpler, since the lemma needs only to give a crushing perturbation of a sequence of linear isomorphisms.

1.4 Structure of the work

In Section 2, we present some basic background which will be used throughout the text. In §2.1, we give a slightly more general definition of Poincaré maps and present a change of coordinates that straightens the local stable and unstable manifolds around a hyperbolic saddle, with the additional property that the Euclidean norm in this coordinate system is adapted, that is, the flow presents immediate hyperbolic contraction (resp., expansion) in the stable (resp., unstable) coordinate. These adapted coordinates are used in the proof of the existence of singular flow boxes around hyperbolic saddles (Lemma 3.0.7). In §2.3, we make some remarks about linear cocycles, specially about one specific cocycle that plays a major role in the proof of our result - the linear Poincaré flow. We give also an example of a nonlinear cocycle - the orthonormal frame flow - which is a main tool in the construction of the tubular chart in Section 4. We have already mentioned that the non-existence of acips is equivalent to a volume crushing property. In §2.5, we state and prove this criterion, with some important observations about volume crushing. In §2.6, we prove a lemma about integrals of functions with bounded logarithmic derivative which is used to proof that, for long tubular neighborhoods, the volume concentrated on the edges are relatively small. As in [AB1], we need to use the Vitali covering theorem to guarantee that, except for a small set, we can cover the manifold with crushable sets. In §2.7, we make a precise definition of Vitali Coverings and state this theorem.

In Section 3, as mentioned above, we prove the existence of a singular flow box around a hyperbolic saddle (Lemma 3.0.7) and use this lemma to construct a transverse section with the property that every point in the manifold not contained in the stable manifold of a sink (resp. the unstable manifold of a source) must hit the section for the future (resp. for the past).

Section 4 is devoted to prove the existence of a C^2 tubular chart which enable the construction of the perturbation in $\mathbb{R}^d$. The chart has several natural properties and some technical ones. Before the proof of Theorem 4.0.15 we give some informal explanation about this properties and how they help us in the construction of the perturbation.

We have already made some remarks about the Fettuccine's Lemma, which gives us a perturbation of the vector field inside a long tubular neighborhood. Besides the tubular chart, the proof of this lemma needs several other ingredients and Section 5 is all devoted to present those tools and proving the Lemma (the proof is given only in § 5.4). In §5.1 we give, in Lemma 5.1.2, an explicit formula for the time $t_0 = t_0(\epsilon, \delta)$ that must elapse for an ϵ-perturbation generate a δ-crushing property. We call this amount of time the *crushing-time*. As in [AB1], the "size" $T > 0$ of the tubular neighborhood which supports the perturbation (in that case, the height n of the open tower) must be much bigger than the crushing-time. The reason is that the end of the crushable set (with size less than t_0) cannot be crushed. However, taking $T \gg t_0$, we guarantee that the relative volume of the non-crushed part is sufficiently small. In §5.2 we define the *sliced tubes*, a type of tubular neighborhood saturated by orthogonal cross-sections which are related to the Linear Poincaré Flow. These sets are convenient to work with for many reasons. They are not bent in the direction of the flow, for example, and their volumes are easily computed. Proposition 5.2.7 shows that we can approximate a standard tubular neighborhood by sliced tubes and in §5.3 we show how to construct a bump function with bounded C^1-norm inside a sliced tube. Finally, in Section 6, we extend the local crushing property to the whole manifold, proving that it is possible to cover the manifold (except for a small set) with crushable sets given by Lemma 5.0.22.

2
Preliminaries

In this section we collect several standard definitions and facts from Linear Algebra, Differential Topology and Dynamical Systems that will be used later.

2.1
Basic facts about vector fields and flows

One important tool in the analysis of the local structure of periodic orbits is the Poincaré first return map, a discrete dynamical system defined in a cross section that inherits local properties of the flow close to a periodic orbit. In this work we use a more general definition of the Poincaré map, which allows the map to be a first hit map between two cross sections. We also allow the cross sections to be general codimension 1 submanifolds with boundary. It is convenient to impose a certain *compatibility* condition on those submanifolds:

Definition 2.1.1 (compatibility) *Given $X \in \mathfrak{X}^1(M)$ and its induced flow $\{\varphi^t\}_t$, we say that two codimension 1 submanifolds with boundary Σ_1 and Σ_2 are* compatible *if their union is still a submanifold with boundary, if they are both transverse to X and the following holds:*

$$\inf\{t > 0 : \varphi^t(x) \in \Sigma_2\} \leq \inf\{t > 0 : \varphi^t(x) \in \Sigma_1\},\ \forall x \in \Sigma_1 \tag{2.1}$$
$$\inf\{t > 0 : \varphi^{-t}(y) \in \Sigma_1\} \leq \inf\{t > 0 : \varphi^{-t}(y) \in \Sigma_2\},\ \forall y \in \Sigma_2. \tag{2.2}$$

So compatibility forbids the situation of Figure 2.1. Also, note that the above definition does not exclude the possibility of Σ_1 being equal to Σ_2, since we want to consider the Poincaré first return map as a particular case of the map we are about to construct.

Definition 2.1.2 (hitting-time) *Let Σ_1 and Σ_2 be compatible cross sections. Then we define the* hitting-time *function $\tau : \Sigma_1 \to \mathbb{R}^+ \cup \{\infty\}$ by*

$$\tau(x) = \inf\{t > 0 : \varphi^t(x) \in \Sigma_2\}$$

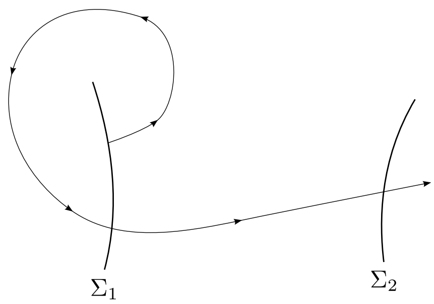

Figure 2.1: Non-compatible cross-sections.

and its backwards version $\tau' : \Sigma_2 \to \mathbb{R}^+ \cup \{\infty\}$ *by*

$$\tau'(y) = \inf\{t > 0 : \varphi^{-t}(y) \in \Sigma_2\}.$$

Here $\tau(x) = \infty$ *(resp.* $\tau'(y) = \infty$*) means that the future orbit of* x *(resp. the past orbit of* y*) does not intersect* Σ_2 *(resp.* Σ_1*).*

Proposition 2.1.3 (Poincaré Map) *Let* Σ_1 *and* Σ_2 *be two compatible cross sections and let* σ_1 *and* σ_2 *be their respective induced Riemannian measures. Consider the following subsets of the cross sections:*

$$\tilde{\Sigma}_1 = \{x \in \Sigma_1 \backslash \partial\Sigma_1 : \tau(x) < \infty,\ \varphi^{\tau(x)}(x) \in \Sigma_2 \backslash \partial\Sigma_2\}$$
$$\tilde{\Sigma}_2 = \{y \in \Sigma_2 \backslash \partial\Sigma_2 : \tau'(y) < \infty,\ \varphi^{-\tau'(y)}(y) \in \Sigma_1 \backslash \partial\Sigma_1\}.$$

Then:

1. $\tilde{\Sigma}_1$ *is open in* Σ_1*;*

2. $\tilde{\Sigma}_2$ *is open in* Σ_2*;*

3. $\tau|_{\tilde{\Sigma}_1}$ *and* $\tau'|_{\tilde{\Sigma}_2}$ *are* C^1 *maps;*

4. *the map*

$$\begin{array}{rccl} f: & \tilde{\Sigma}_1 & \to & \tilde{\Sigma}_2 \\ & x & \mapsto & \varphi^{\tau(x)}(x) \end{array}$$

is a diffeomorphism, with inverse

$$\begin{array}{rccl} f^{-1}: & \tilde{\Sigma}_2 & \to & \tilde{\Sigma}_1 \\ & y & \mapsto & \varphi^{-\tau'(y)}(y) \end{array};$$

5. *for* σ_1*-a.e.* $x \in \Sigma_1$*, either* $\tau(x) = \infty$ *or* $x \in \tilde{\Sigma}_1$*.*

Proof: Parts 1, 2 and 3 are easy consequences of the (long) flow-box theorem (Proposition 1.1 in Chapter 3 of [PdM]). Notice that the compatibility of the cross sections guarantees that f is one-to-one. Its inverse is given by

$$f^{-1}(y) = \varphi^{-\tau'(y)}(y).$$

It follows from part 3 that f and f^{-1} are C^1 maps, thus proving part 4.

For the proof of part 5, define the following subsets:

$$F_i = \bigcup_{t\in\mathbb{R}} \varphi^t(\partial\Sigma_i), \quad (i = 1, 2).$$

Then $F_i \subset M$ is an immersed codimension 1 submanifold transverse to Σ_1. Therefore the intersection $F_i \cap \Sigma_1$ is an immersed codimension 2 submanifold of Σ_1, and in particular it has zero σ_1 measure. Noticing that $x \in \Sigma_1 \setminus (F_1 \cup F_2)$ implies that either $\tau(x) = \infty$ or $x \in \tilde{\Sigma}_1$, the proof of the proposition is concluded. ■

The diffeomorphism $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ defined in the previous proposition will be called *Poincaré map.*

Corollary 2.1.4 *Let Σ_1 and Σ_2 be two compatible cross sections and let $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ be the induced Poincaré map. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset \tilde{\Sigma}_1$ is a measurable set with $\sigma_1(A) < \delta$, then*

$$m\left(\bigcup_{p\in A}\bigcup_{t\in[0,\tau(p)]}\varphi^t(p)\right) < \epsilon.$$

Proof: It suffices to note that

$$\sigma_*(A) = m\left(\bigcup_{p\in A}\bigcup_{t\in[0,\tau(p)]}\varphi^t(p)\right)$$

defines a measure on $\tilde{\Sigma}_1$ which is absolutely continuous with respect to σ_1. ■

Notice the following consequence of long flow-box theorem:

Remark 2.1.5 *Let $t_0 > 0$ and let $p \in M$ be a non-periodic point or a periodic point with period bigger then t_0. Suppose Σ_1 and Σ_2 are cross sections (i.e., codimension 1 submanifolds tranverse to the flow) such that $p \in \Sigma_1 \setminus \partial\Sigma_1$ and $\varphi^{t_0}(p) \in \Sigma_2 \setminus \partial\Sigma_2$. Then there exist closed neighborhoods Σ_1^* and Σ_2^* of p and $\varphi^{t_0}(p)$ in Σ_1 and Σ_2 respectively that are compatible cross-sections. Moreover, $\tau(x) < \infty$ for all $x \in \Sigma_1^*$.*

We say that the Poincaré map as in Remark 2.1.5 is *based* on the orbit of p with respect to the base time t_0 and denote its hitting time by τ_{X,p,t_0}. Depending on the context, the vector field X, the point p and/or the base time t_0 that define the hitting-time map with respect to a segment of orbit will be omitted from the notation, yielding τ_X, τ_{t_0} or simply τ. We denote this Poincaré map by

$$\begin{array}{rrcl} \Phi_{t_0}: & \Sigma_1^* & \to & \Sigma_2^* \\ & x & \mapsto & \Phi_{t_0}(x) = \varphi^{\tau(x)}(x). \end{array} \tag{2.3}$$

Remark 2.1.6 *Since the hitting-time map is C^1 and, therefore, continuous, we have that for all $\epsilon > 0$ there exists a neighborhood $V \subset \Sigma_1^*$ of p such that*

$$|t_0 - \tau_{t_0}(x)| < \epsilon$$

for all $x \in V \cap \Sigma_1^*$.

Recall that if $G : U_1 \to U_2$ is a diffeomorphism and $X \in \mathfrak{X}^1(U_1)$ is a vector field, we define its *push-forward* $F_*X \in \mathfrak{X}^1(U_2)$ by

$$(F_*X)(z) \equiv DG(G^{-1}(z)) \cdot X(G^{-1}(z)).$$

The flows of the two vector fields are conjugate by the diffeomorphism G.

In Section 3, we will consider the Poincaré map with respect to some well-chosen sections with properties. In that construction, we will make use of "adapted" coordinates around a hyperbolic singularity (i.e., a fixed point of the flow), which are given by next lemma. The *stable* (resp. *unstable*) *index* of a hyperbolic singularity is the dimension of its stable (resp. unstable) manifold; in particular the sum of the indices equals $d = \dim M$.

Lemma 2.1.7 (adapted coordinates) *Let $X \in \mathfrak{X}^1(M)$. Suppose $p \in M$ is a hyperbolic singularity of X, and let s and u be respectively the stable and unstable indices. Then there exist*

- *a chart $F : U \to V$, where U and V are open neighborhoods of $p \in M$ and $0 \in \mathbb{R}^d$, respectively;*
- *constants $\Lambda > \lambda > 0$;*

with the following properties:

1. *$F(p) = 0$.*
2. *The local stable (resp. unstable) manifold at p is mapped by F into $\mathbb{R}^s \times \{0\}$ (resp. $\{0\} \times \mathbb{R}^u$).*

3. *Suppose $x : \mathbb{R} \to M$ is an orbit of the flow generated by X and $I \subset \mathbb{R}$ is an interval such that $x(I) \subset U$. For $t \in I$, write*

$$F(x(t)) = \big(y_{\mathrm{s}}(t), y_{\mathrm{u}}(t)\big) \quad \textit{with} \quad y_{\mathrm{s}}(t) \in \mathbb{R}^s,\ y_{\mathrm{u}}(t) \in \mathbb{R}^u\,.$$

Then for all t_0, $t_1 \in I$ with $t_0 < t_1$ we have:

$$e^{-\Lambda(t_1-t_0)}\|y_{\mathrm{s}}(t_0)\| \le \|y_{\mathrm{s}}(t_1)\| \le e^{-\lambda(t_1-t_0)}\|y_{\mathrm{s}}(t_0)\|\,, \tag{2.4}$$
$$e^{\lambda(t_1-t_0)}\|y_{\mathrm{u}}(t_0)\| \le \|y_{\mathrm{u}}(t_1)\| \le e^{\Lambda(t_1-t_0)}\|y_{\mathrm{u}}(t_0)\|\,, \tag{2.5}$$

where $\|\cdot\|$ denotes Euclidean norm.

Lemma 2.1.7 is probably well-known, but being without a precise reference, we will provide a proof. We begin with the following linear algebraic fact:

Lemma 2.1.8 *Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map without purely imaginary eigenvalues. Let E^{s} (resp. E^{u}) be the generalized eigenspace corresponding to eigenvalues of negative (resp. positive) real part. Then there exists an "adapted" inner product $\langle\cdot,\cdot\rangle_{\mathrm{a}}$ on $\mathbb{R}^d$ and constants $\Lambda > \lambda > 0$ such that, for all $v_{\mathrm{s}} \in E^{\mathrm{s}}$, $v_{\mathrm{u}} \in E^{\mathrm{u}}$ we have:*

$$\langle v_{\mathrm{s}}, v_{\mathrm{u}}\rangle_{\mathrm{a}} = 0\,, \tag{2.6}$$
$$-\Lambda\|v_{\mathrm{s}}\|_{\mathrm{a}}^2 \le \langle Lv_{\mathrm{s}}, v_{\mathrm{s}}\rangle_{\mathrm{a}} \le -\lambda\|v_{\mathrm{s}}\|_{\mathrm{a}}^2\,, \tag{2.7}$$
$$\lambda\|v_{\mathrm{u}}\|_{\mathrm{a}}^2 \le \langle Lv_{\mathrm{u}}, v_{\mathrm{u}}\rangle_{\mathrm{a}} \le \Lambda\|v_{\mathrm{u}}\|_{\mathrm{a}}^2\,. \tag{2.8}$$

where $\|v\|_{\mathrm{a}}^2 = \langle v, v\rangle_{\mathrm{a}}$.

Proof: First consider the case where all eigenvalues of L have negative real part. Then the exponential matrix e^L has spectral radius $\rho < 1$. Let $\|\cdot\|$ be the Euclidean norm. By the spectral radius theorem (Gelfand formula), we have $\lim_{t\to+\infty} \frac{1}{t}\log\|e^{tL}\| = \log\rho < 0$. Therefore the following expression defines a new norm:

$$\|v\|_{\mathrm{a}}^2 = \int_0^\infty \|e^{tL}\cdot v\|^2\,dt\,.$$

It is clear that this norm corresponds to an inner product $\langle\cdot,\cdot\rangle_{\mathrm{a}}$. Notice that

$$s \ge 0 \ \Rightarrow\ \|e^{sL}\cdot v\|_{\mathrm{a}}^2 = \int_s^\infty \|e^{tL}\cdot v\|^2\,dt\,.$$

In particular,

$$\left.\frac{d}{ds}\right|_{s=0} \|e^{sL}\cdot v\|_{\mathrm{a}}^2 = -\|v\|^2\,.$$

On the other hand, the same derivative can be computed as

$$\left.\frac{d}{ds}\right|_{s=0} \langle e^{sL}\cdot v, e^{sL}\cdot v\rangle_{\mathrm{a}} = 2\langle Lv, v\rangle_{\mathrm{a}} \,.$$

Thus $\langle Lv, v\rangle_{\mathrm{a}} = -\frac{1}{2}\|v\|^2$, which is between $-\Lambda\|v\|_{\mathrm{a}}^2$ and $-\lambda\|v\|_{\mathrm{a}}^2$ for some constants $\Lambda > \lambda > 0$.

We proved the lemma in the particular case where all eigenvalues of L have negative real part. The general case of the lemma follows by considering the restrictions $L|E^{\mathrm{s}}$ and $(-L)|E^{\mathrm{u}}$ and taking the orthogonal sum inner product. ■

Remark 2.1.9 *All inner products on* $\mathbb{R}^d$ *coincide modulo a linear change of coordinates. Therefore, in the situation of Lemma 2.1.8 we can find an invertible linear map* $S : \mathbb{R}^d \to \mathbb{R}^d$ *such that if* L, E^{s} *and* E^{u} *are replaced with* SLS^{-1}, $S(E^{\mathrm{s}})$, $S(E^{\mathrm{u}})$, *then the relations* (2.6), (2.7), (2.8) *hold with* $\langle\cdot,\cdot\rangle_{\mathrm{a}}$ *being the Euclidean inner product.*

Proof of Lemma 2.1.7: By changing coordinates, we can assume that the vector field X is defined on a neighborhood of $p = 0$ in $\mathbb{R}^d$. Let E^{s} and E^{u} denote the stable and unstable subspaces. As a trivial consequence of the stable manifold theorem (see for example [PdM, pp.88–89]), we can change coordinates again so that the local stable and unstable manifolds are contained in the vector subspaces E^{s} and E^{u}, respectively. By applying the linear change of coordinates given by Remark 2.1.9, we can assume that there are constants $\Lambda > \lambda > 0$ such that relations (2.6), (2.7), (2.8) hold for $L = DX(0)$, with $\langle\cdot,\cdot\rangle_{\mathrm{a}}$ being the Euclidean inner product. By a final change of coordinates using an orthogonal linear map, we can assume that $E^{\mathrm{s}} = \mathbb{R}^s \times \{0\}$ and $E^{\mathrm{u}} = \{0\} \times \mathbb{R}^u$.

In coordinates $(y_{\mathrm{s}}, y_{\mathrm{u}}) \in \mathbb{R}^s \times \mathbb{R}^u$, we write

$$X(y_{\mathrm{s}}, y_{\mathrm{u}}) = \big(X_{\mathrm{s}}(y_{\mathrm{s}}, y_{\mathrm{u}}), X_{\mathrm{u}}(y_{\mathrm{s}}, y_{\mathrm{u}})\big) \,.$$

Then we have

$$X_{\mathrm{s}}(y_{\mathrm{s}}, 0) = 0 \,, \quad X_{\mathrm{u}}(0, y_{\mathrm{u}}) = 0 \,.$$

Fix a positive $\epsilon < \lambda$. Then for every $(y_{\mathrm{s}}, y_{\mathrm{u}})$ sufficiently close to $(0, 0)$, we have

$$\begin{aligned} \big\|X_{\mathrm{s}}(y_{\mathrm{s}}, y_{\mathrm{u}}) - L(y_{\mathrm{s}}, 0)\big\| &\le \epsilon\|y_{\mathrm{s}}\| \,, \\ \big\|X_{\mathrm{u}}(y_{\mathrm{s}}, y_{\mathrm{u}}) - L(0, y_{\mathrm{u}})\big\| &\le \epsilon\|y_{\mathrm{u}}\| \,. \end{aligned}$$

We reduce the chart domain so that these properties are satisfied. Now assume that $t \in I \mapsto (y_{\mathrm{s}}(t), y_{\mathrm{u}}(t))$ is a trajectory of the flow contained in this chart domain. Then

$$\left.\frac{d}{dt}\right|_{t=0} \|y_{\rm s}(t)\|^2 = 2\big\langle X_{\rm s}(y_{\rm s}(t), y_{\rm u}(t)), (y_{\rm s}(t), 0)\big\rangle$$
$$\leq 2(-\lambda + \epsilon)\|y_{\rm s}(t)\|^2$$

This implies that the second inequality in (2.4) holds with $\lambda - \epsilon$ in the place of λ. The remaining inequalities are proven similarly (with Λ replaced by $\Lambda + \epsilon$). ■

Recall that the main part of the proof of the main result is to perturb a given vector field so that it has the δ-crushing property. Actually we will perform a few successive perturbations, each one preparing the ground for the next one. In this regard, the following fact will be useful:

Proposition 2.1.10 *The set $\mathfrak{I} \subset \mathfrak{X}^r(M)$ of vector fields such that all periodic orbits are hyperbolic (and isolated) is a C^1-open and dense set.*

Proof: This proposition is a intermediate step of the proof of the Kupka–Smale Theorem and can be found for example in [PdM, p.115]. ■

2.2 Non-Conformality

If L is a linear isomorphism between inner-product vector spaces, the *non-conformality* of L is

$$\mathrm{NC}(L) \equiv \|L\|\,\|L^{-1}\|.$$

This quantity measures how much L can distort angles, in fact:

$$\frac{1}{\|L\| \cdot \|L^{-1}\|} \leq \frac{\sin(\angle(Lu, Lv))}{\sin(\angle(u, v))} \leq \|L\| \cdot \|L^{-1}\|. \tag{2.9}$$

See [BV, Lemma 2.7] for a proof of (2.9).

Next Proposition is a simple Linear Algebra fact and follows from the definition of matrix induced norm (See Figure 2.2 for an illustrative idea of the proof).

Proposition 2.2.1 *Let L be an invertible map and let $\mathcal{B}(r)$ be the Euclidean ball with radius r centered in the origin. If $r_1 < r_2$ are such that*

$$B(r_1) \subset L(B(r)) \subset B(r_2), \tag{2.10}$$

then $r_2 > r_1 \cdot NC(L)$. Moreover, $r_2 > r \cdot \max\{\|L\|, \|L^{-1}\|\}$ satisfies (2.10), with $r_2 = r_1 \cdot \mathrm{NC}(L)$.

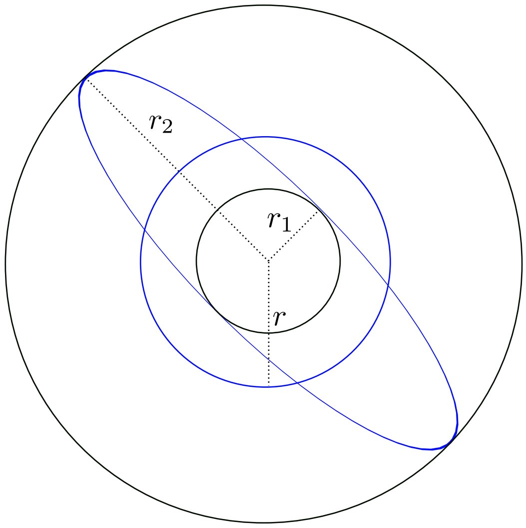

Figure 2.2: The image of an Euclidean ball by a linear invertible map L is incribed in a sphere with radius $r_2 = \|L\|r$ and circumscribed on a sphere with radius $r_1 = \|L^{-1}\|^{-1}r$.

2.3 Linear Cocycles

The word "cocycle" can be found in Mathematics with very different meanings and the term seems to have been borrowed from Algebraic Topology. Let us see its Dynamical Systems' definition.

Definition 2.3.1 *A* flow *on a manifold M is an action of $\mathbb{R}$ by diffeomorphisms, i.e., a collection of diffeomorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$ such that $\varphi^{t+s} = \varphi^t \circ \varphi^s$. We also ask the joint map $(t, x) \in \mathbb{R} \times M \mapsto \varphi^t(x) \in M$ to be continuous.*

Definition 2.3.2 *Let $\varphi^t : M \to M$ be a flow on a smooth manifold M and let $\pi : E \to M$ be a fiber bundle over M. A* cocycle *over the flow φ^t is a flow*

$$F^t : E \to E$$

such that $\pi \circ F^t = \varphi^t \circ \pi$.

Notice that the restriction of F^t to the fiber $\pi^{-1}(x)$ is a diffeomorphism onto the fiber $\pi^{-1}(\varphi^t x)$, which we denote by $A^t(x) : \pi^{-1}(x) \to \pi^{-1}(\varphi^t x)$. The following properties hold:

1. $A^0(x) = Id$;

2. $A^{t+s}(x) = A^s(\varphi^t(x))A^t(x)$. (cocycle condition).

A special case is that of *linear cocycles*:

Definition 2.3.3 *Let $\varphi^t : M \to M$ be a flow on a smooth manifold M. Let $\pi : E \to M$ be a vector bundle over M. A cocycle $F^t : E \to E$ is called a* linear cocycle *if the maps $A^t(x)$ between fibers are linear.*

In the case the vector bundle E is trivial, i.e., $E = M \times \mathbb{R}^n$, then the linear cocycle takes the form:

$$F^t(x, v) = (\varphi^t(x), A^t(x)v),$$

where $A^t(x) \in GL(n, \mathbb{R})$ for all $x \in M$. Conversely, if A^t is a family of linear maps with $A^0 = Id$ and satisfying the cocycle condition then we can define a linear cocycle by the formula above. The family of linear maps $A^t : M \to GL(n, \mathbb{R})$ will be ambiguously called "cocycle".

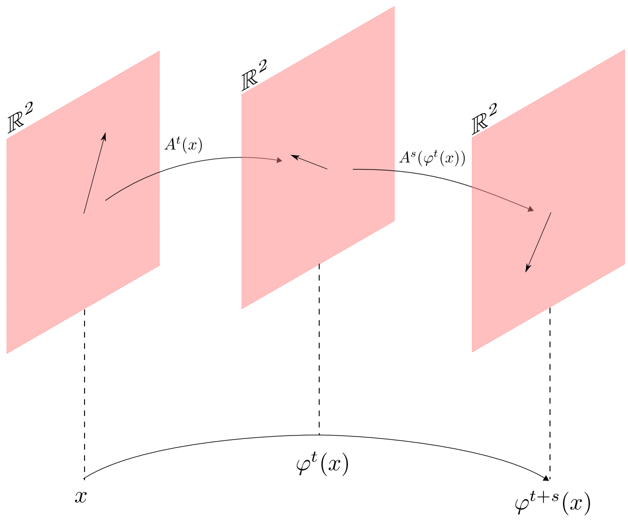

Figure 2.3: A linear cocycle over the flow $\{\varphi^t\}$.

Definition 2.3.4 *Let $A^t : M \to GL(n, \mathbb{R})$ be a cocycle which is differentiable in the t parameter. The (infinitesimal) generator of A^t is the function $a : M \to GL(n, \mathbb{R})$, given by*

$$a(x) = \frac{\partial}{\partial t} A^t(x)\bigg|_{t=0}.$$

Remark 2.3.5 *The name generator in the previous definition comes from the fact that a cocycle may be generated by a non-autonomous differential equation:*

$$\frac{\partial}{\partial t} A^t(x) = a(\varphi^t(x)) A^t(x),$$

with initial condition $A^0(x) = Id$.

Proposition 2.3.6 *Let* $A^t : M \to GL(n, \mathbb{R})$ *be a cocycle with generator* $a : M \to GL(n, \mathbb{R})$*. Then we have:*

1. $\|A^t(p)\| \leq e^{C|t|}$;
2. $\|A^t(p) - Id\| \leq e^{C|t|} - 1$,

where $C = \sup_{x \in M} \|a(x)\|$.

Proof: In order to prove part 1, define $f(t) = \|A^t(x)\|$ and note that

$$\begin{aligned} |f'(t)| &\leq \left\| \frac{\partial}{\partial t} A^t(x) \right\| \\ &= \|a(\varphi^t(x)) A^t(x)\| \\ &\leq C f(t). \end{aligned}$$

That is, $|(\log f(t))'| \leq C$. Since $f(0) = \|Id\| = 1$, we have $f(t) \leq e^{C|t|}$, as we wanted to show.

The proof of part 2 is analogous. Let us now consider $B^t = A^t - Id$. Then we have

$$\frac{\partial}{\partial t} B^t(x) = a(\varphi^t(x))(Id + B^t(x)).$$

Defining the function $g(t) = \|B^t(x)\|$, we have that

$$\begin{aligned} |g'(t)| &\leq \left\| \frac{\partial}{\partial t} B^t(x) \right\| \\ &\leq C(1 + g(t)). \end{aligned}$$

The solution of the ODE:

$$\begin{aligned} &h'(t) = C(1 + h(t)), \\ &h(0) = 0, \end{aligned}$$

is $h(t) = e^{Ct} - 1$. Thus if $t > 0$ then $g(t) \leq \int_0^t |g'| \leq h(t) = e^{Ct} - 1$, and analogously for $t < 0$. ∎

Every linear cocycle in $GL(n, \mathbb{R})$ over a flow φ^t induces a cocycle in $\mathbb{R}$, by taking the determinant of the matrix $A^t(x)$. The precise statement of this well known result is given by the following proposition and its proof can be found for example in [CL, Theorem I.7.3].

Proposition 2.3.7 *Let $A : \mathbb{R} \times M \to GL(n, \mathbb{R})$ be a cocycle over the flow $\varphi : \mathbb{R} \times M \to M$ with generator $G : M \to GL(n, \mathbb{R})$. Then the function $f : \mathbb{R} \times M \to \mathbb{R}$, defined by*

$$f(t, p) = \det A^t(p)$$

is a linear cocycle in $\mathbb{R}$ over the same flow. Moreover its generator $g : M \to \mathbb{R}$ is given by

$$g(p) = \operatorname{tr} G(p). \tag{2.11}$$

Consider $X \in \mathfrak{X}^1(M)$. Let us see some natural examples of linear cocycles over the flow φ^t generated by a vector field X. The first one is the *derivative cocycle*:

$$T_x M \to T_{\varphi^t(x)} M$$

$$u \mapsto D\varphi^t(x)u,$$

The cocycle condition is a direct consequence of the chain rule.

The second example is the *linear Poincaré flow*. Let $R(X) \subset M$ be the set of regular points in M, that is,

$$R(X) = \{x \in M : X(x) \neq 0\}.$$

Let us define the normal bundle $N_{R(X)}$ associated to X. For each $x \in R(X)$, let N_x be the orthogonal complement of $X(x)$ in $T_x M$. This is a fiber of a vector bundle over $R(X)$, which is a subbundle of $T_{R(X)} M$.

Definition 2.3.8 *The* linear Poincaré flow *of X is defined over $N_{R(X)}$ by*

$$P^t_x : N_x \to N_{\varphi^t(x)}$$

$$u \mapsto \Pi_{\varphi^t(x)} \circ D\varphi^t(x)u,$$

where $\Pi_x : T_x M \to N_x$ denotes the orthogonal projection on the normal subbundle.

The cocycle condition of the linear Poincaré flow follows from the chain rule.

The linear Poincaré flow is commonly used in the study of flows local behavior; the reason is given by the next proposition.

Proposition 2.3.9 *Let $\Sigma_1 \ni p$ and $\Sigma_2 \ni \varphi^t(p)$ be two cross sections, let $\Phi_t : \Sigma_1 \to \Sigma_2$ be the Poincaré map based on the orbit of p, and let $P_p^t : N_p \to N_{\varphi^t(p)}$ be a map from the linear Poincaré flow. Then the following diagram commutes:*

$$\begin{tikzcd} T_p\Sigma_1 \arrow[r, "D\Phi_t(p)"] \arrow[d, "\Pi_p"'] & T_{\varphi^t(p)}\Sigma_2 \arrow[d, "\Pi_{\varphi^t(p)}"] \\ N_p \arrow[r, "P_p^t"'] & N_{\varphi^t(p)} \end{tikzcd}$$

In particular, if $X(p) \perp T_p\Sigma_1$ and $X(\varphi^t(p)) \perp T_{\varphi^t(p)}\Sigma_2$ then

$$D\Phi_t(p) = P_p^t.$$

Proof: Fix $u \in T_p\Sigma_1$. We have $\Phi_t(x) = \varphi^{\tau(x)}(x)$, where τ is the hitting-time. Differentiating, we obtain

$$D\Phi_t(p) \cdot u = D\varphi^t(p) \cdot u + (D\tau(p) \cdot u)X(\varphi^t p).$$

Write $u = \Pi_p(u) + cX(p)$; then

$$D\Phi_t(p) \cdot u = D\varphi^t(p) \circ \Pi_p(u) + (c + D\tau(p) \cdot u)X(\varphi^t p).$$

Since $\Pi_{\varphi^t p}(X(\varphi^t p)) = 0$, we have

$$\begin{aligned} \Pi_{\varphi^t p} \circ D\Phi_t(p) \cdot u &= \Pi_{\varphi^t p} \circ D\varphi^t(p) \circ \Pi_p(u) \\ &= P_p^t \circ \Pi_p(u), \end{aligned}$$

as we wanted to show. ■

We will often deal with the linear Poincaré flow based on a segment of the orbit of a point p. In this case we will use the following notation:

$$\begin{array}{rccl} P_p^{s,t}: & N_{\varphi^s(p)} & \to & N_{\varphi^t(p)} \\ & u & \mapsto & \Pi_{\varphi^t(p)} \circ D\varphi^{t-s}(\varphi^s(p))u. \end{array}$$

In this notation we include the possibility of $t < s$. So, as a consequence of the cocycle condition, we obtain $(P_p^{t,s})^{-1} = P_p^{s,t}$. In Section 5, the initial base-point will be $0 \in \mathbb{R}^{d-1}$, so we omit it from the notation, yielding $P_s^t = P_0^{s,t}$.

An example of a natural and useful non-linear cocycle appears in the next subsection.

2.4
The orthonormal frame flow

In Section 4, we will define a tubular chart with some useful geometrical properties. To construct this chart, a bundle structure is necessary – the orthonormal frame bundle. We will also need to define a special cocycle over this bundle – the orthonormal frame flow.

Recall that M is a smooth (C^∞) compact manifold of dimension d, endowed with a Riemannian metric. For each $x \in M$, let $\mathfrak{F}_x$ be the set of *orthonormal frames* on the tangent space T_xM (i.e. ordered orthonormal bases of T_xM). Let $\mathfrak{F} = \bigsqcup_{x\in M} \mathfrak{F}_x$. One can define a smooth differentiable structure on $\mathfrak{F}$ so that the obvious projection $\Pi : \mathfrak{F} \to M$ is smooth and defines a fiber bundle, whose fibers are diffeomorphic to the orthonormal group $\mathrm{O}(d)$. This is called the *orthonormal frame bundle* of M.

There is an equivalent way of constructing this bundle: An *oriented flag* at the point $x \in M$ is a nested sequence $F_1 \subset F_2 \subset \cdots \subset F_d$ of vector subspaces of T_xM with $\dim F_i = i$. Given such an oriented flag, there exists an orthonormal frame $(e_1, \ldots, e_d)$ such that F_i is spanned by $e_1, \ldots, e_i$. This correspondence is one-to-one and onto. Therefore $\mathfrak{F}$ can also be viewed as a bundle of oriented flags.

Next, fix a vector field $X \in \mathfrak{X}^r(M)$, and let $\{\varphi^t\}_t$ be the induced flow on M. Then we define a flow on $\mathfrak{F}$ as follows: For each $t \in \mathbb{R}$, the t-image of the orthonormal frame $(e_1, \ldots, e_d) \in \mathfrak{F}_x$ is obtained by applying the Gram–Schmidt process to the frame $(D\varphi^t(x) \cdot e_1, \ldots, D\varphi^t(x) \cdot e_d)$. This is called the *orthonormal frame flow.* It is a flow of class C^{r-1}.

Using the identification between orthonormal frames and oriented flags, the orthonormal frame flow can be described as follows: for each $t \in \mathbb{R}$, the t-image of the flag $F_1 \subset F_2 \subset \cdots \subset F_d = T_xM$ is the flag $D\varphi^t(x)(F_1) \subset D\varphi^t(x)(F_2) \subset \cdots \subset D\varphi^t(x)(F_d)$, where each space is endowed with the induced orientation.

Remark 2.4.1 *More generally, given any vector bundle endowed with a Riemannian metric, one can define an associated orthonormal frame bundle, and given a linear cocycle on the vector bundle, one can define an associated orthonormal frame flow. We will not need those more general constructions.*

2.5 Basic facts about volume crushing

There is a somewhat philosophical obstacle in trying to prove, in a direct way, a theorem of nonexistence. In order to circumvent such issue we present in this section a lemma that reduces our problem to an existence one. It is merely a version for flows of [AB1, Lemma 1].

Lemma 2.5.1 (Criterion for non-existence of acip) *A flow $\{\varphi^t\}$ generated by a vector field $X \in \mathfrak{X}^1(M)$ has no acip iff for every $\epsilon > 0$ there exists a Borel set $K \subset M$ and $T \in \mathbb{R}$ such that*

$$m(K) > 1 - \epsilon \quad \text{and} \quad m(\varphi^T(K)) < \epsilon.$$

Proof: Notice that the validity of the lemma is unchanged if we replace "$T \in \mathbb{R}$" by "$T \in \mathbb{R}_+$" (just replace K by $M \setminus K$), or by "$T \in \mathbb{N}$" (because the flow up to time 1 cannot distort volumes by more than some constant factor).

We will derive the lemma for the discrete-time version ([AB1, Lemma 1]), which says that a C^1 map $f : M \to M$ has no acip iff for every $\epsilon > 0$ there exists a compact set $K \subset M$ and $T \in \mathbb{N}$ such that

$$m(K) > 1 - \epsilon \quad \text{and} \quad m(f^T(K)) < \epsilon.$$

(Compactness is useful to guarantee measurability of $f^T(K)$ even when f is not invertible.) Notice that if we assume that f is a diffeomorphism, then using the regularity of the measure m, we can replace "compact set" by "Borel set" above.

Notice that a flow $\{\varphi^t\}$ has an acip iff its time-one map φ^1 has an acip; indeed, if μ is an acip for φ^1 then $\bar{\mu} = \int_0^1 \varphi^t_* \mu \, dt$ is an acip for the flow. Hence the lemma follows. ■

For some trivial parts of the dynamics, the crushing property is automatic; for example:

Remark 2.5.2 *Let $X \in \mathfrak{X}^1(M)$. Let M_S be the union all stable manifolds of (hyperbolic) sinks and unstable manifolds of (hyperbolic) sources. If M_S is non-empty then for all $\epsilon > 0$, there is a Borel set $K \subset M_S$ and $T > 0$ such that*

$$m(K) > m(M_S) - \epsilon \qquad \text{and} \qquad m(\varphi^t(K)) < \epsilon \text{ for all } t > T.$$

Proof: Take a small neighborhood V_1 (resp. V_2) of the set of sinks (resp. sources), choose T large, and define $K = \varphi^{-T}(V_1) \cup \varphi^T(M \setminus V_2)$. ■

Later on, our perturbations will be supported on the complement of M_S, because in M_S there is nothing to do.

Remark 2.5.3 *One could improve Remark 2.5.2 by including in M_S also the stable (resp. unstable) sets of the hyperbolic attracting (resp. repelling) periodic orbits of X. For example, if the flow is Morse–Smale then the enlarged M_S has full Lebesgue measure, and it follows from Lemma 2.5.1 that there is no acip. (Of course, this also follows directly from the Poincaré Recurrence Theorem and the fact that the recurrent set for a Morse–Smale flow consists of a finite number of periodic orbits.)*

As explained in the Introduction, the following property is essential to our strategy:

Remark 2.5.4 *For each $\epsilon > 0$, the set*

$$\mathcal{V}_\epsilon = \big\{X \in \mathfrak{X}^1(M) : \textit{there exist a Borel set } K \subset M \textit{ and } T \in \mathbb{R} \textit{ such that } m(K) > 1-\epsilon \textit{ and } m(\varphi_X^T(K)) < \epsilon\big\}$$

is open in the C^1 topology.

Proof: Let $X \in \mathcal{V}_\epsilon$. Take a Borel set $K \subset M$ and $T \in \mathbb{R}$ such that $m(K) > 1-\epsilon$ and $m(\varphi_X^T(K)) < \epsilon$. Choose a positive $\gamma < \epsilon - m(\varphi_X^T(K))$. Take $Y \in \mathfrak{X}^1(M)$ sufficiently C^1-close to X such that

$$|\det(D\varphi_X^T(p)) - \det(D\varphi_Y^T(p))| < \frac{\gamma}{m(K)},$$

for all $p \in M$. Then we obtain that

$$\begin{aligned} m(\varphi_Y^T(K)) &= \int_K |\det(D\varphi_Y^T(p))| dm(p) \\ &< \int_K \left(|\det(D\varphi_X^T(p))| + \frac{\gamma}{m(K)}\right) dm(p) \\ &= m(\varphi_X^T(K)) + \gamma < \epsilon. \end{aligned}$$

And we conclude that $Y \in \mathcal{V}_\epsilon$. ∎

Lemma 2.5.1 and Remark 2.5.4 together imply that that the non-existence of acip is a G_δ property.

2.6
Functions with bounded logarithmic derivative

Recall that the logarithmic derivative of a positive function $f(s)$ is $(\log(f(s)))' = f'(s)/f(s)$.

A simple consequence of the boundedness of the logarithmic derivative is that, in this case, the function presents sub-exponential growth.

Remark 2.6.1 (Sub-exponential growth) *Let $b > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a positive function such that*

$$\left|\frac{d}{ds}(\log(f(s)))\right| < b, \ \ \forall s \in \mathbb{R}.$$

Then

$$e^{-b|s|} < f(s) < e^{b|s|}, \ \ \forall s \in \mathbb{R}.$$

Let $I = [\alpha, \beta] \subset \mathbb{R}$ be a compact interval and let $a > \beta - \alpha$. We will use the following notation:

$$I_a \equiv [\alpha + a, \beta - a] \qquad \text{and} \qquad I^a \equiv [\alpha - a, \beta + a].$$

Proposition 2.6.2 *Given $b > 0$, $t_0 > 0$ and $\gamma \in (0, 1)$, there exists $a_0 > 0$ such that for all $0 < a < a_0$, for any interval I with $|I| > t_0$ and for all positive $f \in C^1(\mathbb{R}, \mathbb{R})$ such that*

$$|f'(s)| \leq bf(s), \quad \forall s \in \mathbb{R}$$

the following holds:

$$\int_{I_a} f(s)ds > (1 - \gamma) \int_{I^a} f(s)ds.$$

Before proving this proposition, we need a lemma:

Lemma 2.6.3 *Let f and b be as in the previous Proposition. Then given $\alpha < \beta$, we have that*

$$b^{-1} \max\{f(\alpha), f(\beta)\}(1-e^{-b(\beta-\alpha)}) < \int_\alpha^\beta f(s)ds < b^{-1} \min\{f(\alpha), f(\beta)\}(e^{b(\beta-\alpha)}-1).$$

Proof: Let $\alpha < t < \beta$. By the hypothesis' inequality,

$$b > \left|\frac{f'(s)}{f(s)}\right|, \tag{2.12}$$

for all $s \in \mathbb{R}$. Integrating both sides from α to t, we obtain

$$\begin{aligned} b(t-\alpha) &> \int_{\alpha}^{t} \left| \frac{f'(s)}{f(s)} \right| ds \\ &> \left| \int_{\alpha}^{t} \frac{f'(s)}{f(s)} ds \right| \\ &= \left| \log \left(\frac{f(t)}{f(\alpha)} \right) \right|. \end{aligned}$$

Which leads us to

$$f(\alpha)e^{-b(t-\alpha)} < f(t) < f(\alpha)e^{b(t-\alpha)}. \tag{2.13}$$

If we integrate both sides of (2.12) from t to β we will obtain a similar conclusion:

$$f(\beta)e^{-b(\beta-t)} < f(t) < f(\beta)e^{b(\beta-t)}. \tag{2.14}$$

Using the righthand side of both (2.13) and (2.14) we conclude that

$$\int_{\alpha}^{\beta} f(t)dt < b^{-1} \min\{f(\alpha), f(\beta)\}(e^{b(\beta-\alpha)} - 1).$$

The same way, using the lefthand side of (2.13) and (2.14) we get

$$\int_{\alpha}^{\beta} f(t)dt > b^{-1} \max\{f(\alpha), f(\beta)\}(1 - e^{-b(\beta-\alpha)}).$$

■

Proof of Proposition 2.6.2: Let

$$a_0 = \min\left\{\frac{t_0}{2}, (2b)^{-1} \log\left(\gamma\frac{(1-e^{-bt_0})}{2} + 1\right)\right\}$$

and assume that $I = [\alpha, \beta]$, with $|\beta - \alpha| < t_0$. Take $0 < a < a_0$ and $T > t_0$ and denote

$$A = \int_{\alpha-a}^{\beta+a} f(s)ds.$$

Our goal is to prove that

$$\int_{\alpha-a}^{\alpha+a} f(s)ds < (\gamma/2)A \quad \text{and} \quad \int_{\beta-a}^{\beta+a} f(s)ds < (\gamma/2)A.$$

Since the proofs of both inequalities are totally analogous, we present only the the proof of the first one.

From Proposition 2.6.2 and the fact that $|\beta - \alpha| < t_0$ we have that

$$A \geq b^{-1} \max\{f(\alpha - a), f(\beta + a)\}(1 - e^{-b(t_0+2a)})$$

$$\geq b^{-1} f(\alpha - a)(1 - e^{-b(t_0+2a)}). \tag{2.15}$$

Again from Proposition 2.6.2, we obtain

$$\int_{\alpha-a}^{\alpha+a} f(s)ds \leq b^{-1} \min\{f(\alpha - a), f(\alpha + a)\}(e^{2ba} - 1)$$

$$\leq b^{-1} f(\alpha - a)(e^{2ba} - 1). \tag{2.16}$$

From Inequalities (2.15), (2.16) and the fact that $0 < a < a_0$, we conclude that

$$\begin{aligned}\int_{\alpha-a}^{\alpha+a} f(s)ds &\leq \frac{A(e^{2ba} - 1)}{1 - e^{-b(t_0+2a)}} \\ &< \frac{A(e^{2ba} - 1)}{1 - e^{-bt_0}} \\ &< A\frac{\exp(\log(\frac{\gamma(1-e^{-bt_0})}{2} + 1)) - 1}{1 - e^{-bt_0}} \\ &= A\frac{\frac{\gamma(1-e^{-bt_0})}{2}}{1 - e^{-bt_0}} \\ &= A\frac{\gamma}{2}.\end{aligned}$$

■

2.7 Vitali Covering

In this work, we use a version of the Vitali Covering Theorem (usually stated in $\mathbb{R}^d$) for compact Riemannian manifolds and include the possibility of the sets in the covering not being balls for the Riemannian metric. For the Theorem still hold in this more general setting, we need that the sets in the cover satisfy a *roundness* property. Roughly speaking, this property means that the sets can be sandwiched by balls for which the ratio between the radii is uniformly bounded. This property is defined in [P, Appendix E].

Definition 2.7.1 (Quasi-roundness) *Let M be a Riemannian Manifold and $x \in M$. We say that $U \subset M$ is a K-quasi-round neighborhood of x if there* *exists $r > 0$ (lower then the injective radius) such that*

$$B_{K^{-1}r}(x) \subset U \subset B_r(x),$$

where $B_r(x)$ is the Riemannian ball around x with radius r.

Definition 2.7.2 (Vitali Cover) *Let $S \subset M$ and let $K > 1$. If $\mathcal{V} = \{V_\alpha\}$ is a cover of S such that for m-a.e. $x \in S$ and for all $r \in (0, \sup_\alpha \operatorname{diam}(U_\alpha))$ there exists a K-quasi-round neighborhood $U \subset \mathcal{V}$ of x with $U \subset B_r(x)$, then we say that $\mathcal{V}$ is a* Vitali Cover of S.

Theorem 2.7.3 (Vitali Covering Theorem) *If $\mathcal{V}$ is a Vitali cover of S, then there exists a countable pairwise disjoint family $\{V_j\}_j \subset \mathcal{V}$ such that*

$$m\left(S \backslash \bigcup_j V_j\right) = 0.$$

Proposition 2.7.4 *Let M and N be compact Riemannian manifolds and let $F : U \subset M \to F(U) \subset N$ be a diffeomorphism with uniform bounded non-conformality, that is, there exists $C > 1$ such that*

$$NC(DF(p)) < C, \quad \forall p \in U.$$

Then for all $x \in U$, there exists $r > 0$ such that for all K-quasi-round neighborhood $V \ni x$ with $\operatorname{diam}(V) < r$*, $F(V)$ is a KC-quasi-round neighborhood of $F(x)$.*

Proof: Since we can take V arbitrarily small, the proof follows from Proposition 2.2.1. ■

Remark 2.7.5 *We conclude, by the previous Proposition and Theorem 2.7.3, that Vitali Covers are preserved by diffeomorphisms with uniform bounded non-conformality.*

3
Transverse Section

In this section, we show that for a C^1 open and dense subset of $\mathfrak{X}^1(M)$, we can construct a transverse section and a return map with some properties (Lemma 3.0.6) that will permit us to use, in Section 6, a non-invariant Rokhlin lemma (Lemma 6.0.2) to obtain a disjoint finite union of tubular neighborhoods that cover M, except for a set of negligible Lebesgue measure.

Recall that a *cross-section* for a flow is a codimension 1 closed submanifold with boundary that is transverse to the vector filed.

Lemma 3.0.6 *Let $X \in \mathfrak{X}^1(M)$ be a vector field with only hyperbolic singularities. Then there exists a cross-section $\Sigma \subset M$ such that:*

1. *if $x \in M$ does not belong to a stable manifold of a sink or saddle singularity then the future orbit of x hits Σ;*
2. *if $x \in M$ does not belong to an unstable manifold of a source or saddle singularity then the past orbit of x hits Σ.*

Before showing how to construct the cross-section Σ, we will prove an intermediate step, which gives the appropriate cross-sections in the neighborhood of a saddle-type singularity.

Lemma 3.0.7 (singular flow-box) *Let p be a hyperbolic singularity of $X \in \mathfrak{X}^1(M)$ of saddle type. Then there exist compatible cross-sections Σ^u and Σ^s with the following properties:*

1. *If $f : \tilde{\Sigma}^u \to \tilde{\Sigma}^s$ is the Poincaré map given by Proposition 2.1.3, then*

$$\tilde{\Sigma}^u = \Sigma^u \setminus \partial\Sigma^u \setminus W^s_{\text{loc}}(p),$$
$$\tilde{\Sigma}^s = \Sigma^s \setminus \partial\Sigma^s \setminus W^u_{\text{loc}}(p).$$

2. *Letting τ be the associated hitting-time, the set*

$$V = \overline{\bigcup_{x \in \tilde{\Sigma}^u} \bigcup_{t \in [0,\tau(x)]} \varphi^t(x)}, \tag{3.1}$$

is a closed neighborhood of the saddle p.

3. *For any point $x \in M \setminus V$, if the future (resp. past) orbit of x hits V then the first hit is in Σ^u (resp. Σ^s).*

See Figure 3.1.

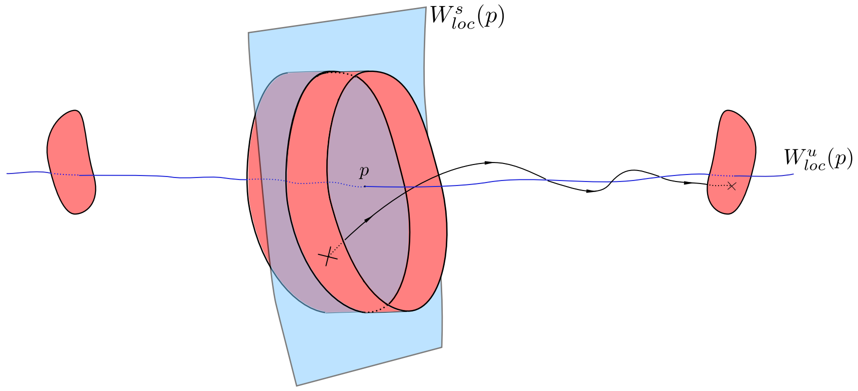

Figure 3.1: A saddle p with $\dim W^s_{loc}(p) = 2$ and $\dim W^u_{loc}(p) = 1$; the cross-sections Σ^u and Σ^s are respectively a cylinder and a union of two disks.

Proof: Let (F, U) be the adapted chart given by Lemma 2.1.7 and r_1, $r_2 > 0$ such that $\overline{B_{r_1}(0)} \times \overline{B_{r_2}(0)} \subset F(U)$. For simplicity of notation, we will work with the adapted coordinates without mentioning the chart F; with abuse of notation, $\{\varphi^t\}$ will denote the flow of the vector field $F_*(X|U)$ on the domain $F(U)$ (therefore not defined for all $t \in \mathbb{R}$).

For $\rho > 0$ sufficiently small, define the following subsets of $\mathbb{R}^d \equiv \mathbb{R}^s \times \mathbb{R}^u$:

$$C^u_\rho \equiv \{x = (x^s, x^u) : \|x^s\| = r_1,\ \|x^u\| \leq \rho\}, \qquad \hat{C}^u_\rho \equiv \{x = (x^s, x^u) \in C^u_\rho : x^u \neq 0\},$$

$$C^s_\rho \equiv \{x = (x^s, x^u) : \|x^s\| \leq \rho,\ \|x^u\| = r_2\}, \qquad \hat{C}^s_\rho \equiv \{x = (x^s, x^u) \in C^s_\rho : x^s \neq 0\}.$$

Claim 3.0.8 *For any $\epsilon \in (0, r_1)$, if $\delta \in (0, r_2)$ is sufficiently small then the future orbit of every point $x \in \hat{C}^u_\delta$ leaves the chart neighborhood without returning to C^u_δ, hitting C^s_ϵ along the way.*

Proof of the Claim: Let $\Lambda > \lambda > 0$ be the constants given by Lemma 2.1.7. Given $\epsilon \in (0, r_1)$, take any $\delta \in (0, r_2)$ such that

$$\left(\frac{\delta}{r_2}\right)^{\lambda/\Lambda} < \frac{\epsilon}{r_1}.$$

Fix a point $x = (x^s, x^u) \in \hat{C}^u_\delta$ and denote its trajectory under the flow by $(x^s(t), x^u(t))$. The norm inequalities from Lemma 2.1.7 hold until the orbit leaves the neighborhood $\overline{B_{r_1}(0)} \times \overline{B_{r_2}(0)}$, i.e., either $\|x^s(t)\| = r_1$ or $\|x^u(t)\| = r_2$. Since $\|x^s(t)\|$ decreases and $\|x^u(t)\|$ exponentially increases with

t, there exists $T > 0$ such that $\|x^u(T)\| = r_2$. Using (2.5) in Lemma 2.1.7, we have

$$r_2 = \|x^u(T)\| \leq e^{\Lambda T} \cdot \|x^u(0)\| = e^{\Lambda T}\delta,$$

which leads us to

$$T > \frac{1}{\Lambda} \log\left(\frac{r_2}{\delta}\right).$$

From the choice of δ, we obtain

$$T > \frac{1}{\Lambda} \log\left(\frac{r_2}{\delta}\right) > \frac{1}{\lambda} \log\left(\frac{r_1}{\epsilon}\right).$$

So, using (2.4) in Lemma 2.1.7, we have

$$\|x^s(T)\| \leq e^{-\lambda T}\|x^s(0)\| = e^{-\lambda T} r_1 < \epsilon$$

Therefore $\varphi^T(x) \in C^s_\epsilon$. This proves the claim. ■

We now continue with the proof of the lemma. Fix any $\epsilon \in (0, r_1)$ and let $\delta \in (0, r_2)$ be given by the claim. By Proposition 2.1.3, there is a Poincaré map $f_+ : \hat{C}^u_\delta \to C^s_\epsilon$ which is a diffeomorphism onto its image.

By symmetry, the claim above also applies to the inverse flow. Therefore we can find some $\epsilon' \in (0, \epsilon)$ (depending on δ) such that the past orbit of every point in $\hat{C}^s_{\epsilon'}$ leaves the chart neighborhood without returning to $C^s_{\epsilon'}$, hitting C^u_δ along the way. By Proposition 2.1.3, there is a Poincaré map $f_- : \hat{C}^s_{\epsilon'} \to C^u_\delta$ which is a diffeomorphism onto its image. Clearly, f_- is a restriction of $(f_+)^{-1}$.

Define $\Sigma^s = C^s_{\epsilon'}$ and $\Sigma_u = \overline{f_-(\hat{C}^s_{\epsilon'})}$. Then Σ_u and Σ_s are compatible cross-sections with the required properties. ■

Remark 3.0.9 *If one assumes that the flow to be smoothly linearizable in a neighborhood of the saddle, then one can slightly simplify the proof of Lemma 3.0.7. By Sternberg Linearization Theorem, that assumption holds for a dense subset of vector fields. However, we preferred to keep things more elementary and avoid linearizations.*

Proof of Lemma 3.0.6: For each point $p \in M$, we define a closed neighborhood $V(p)$ of p and a closed codimension 1 submanifold $\Sigma(p)$ contained in $V(p)$ as follows:

– If p is a saddle-type singularity of X, then apply Lemma 3.0.7 and let $V(p) = V$ and $\Sigma(p) = \Sigma^u \cup \Sigma^s$.

– If p is a sink (resp. source) singularity, let $V(p)$ be a closed ball inside the stable (resp. unstable) manifold of p, whose boundary is a sphere $\Sigma(p)$ transverse to X.

– If $p \in M$ is a non-singular point, let $V(p)$ be a flow-box around p (i.e., a domain given by the flow-box theorem). Let $\Sigma(p)$ be the union of the two "lids" of the flow-box.

Cover the manifold by a finite number of sets $\operatorname{int} V(p)$, and let Σ be the union of the corresponding $\Sigma(p)$. We can arrange that this union is disjoint, and therefore a manifold with boundary. Then Σ is a cross-section with the desired properties. This proves the lemma. ∎

Let Σ be the cross-section given by Lemma 3.0.6. Once we have constructed this transverse section, we need to know how to reduce the study of the dynamics on the manifold to the study of the discrete dynamics on the Poincaré section. Some remarks and propositions in this Section will help answering this question, but it will be totally clear only in Section 6, with Lemma 6.0.2.

Applying Proposition 2.1.3, we obtain subsets $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2 \subset \Sigma$ and a Poincaré map $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$. Let σ be the $(d-1)$-dimensional Riemannian volume on Σ.

Let us introduce some notation that will be used not only in the proof of the following remark but also in Section 6. If $A \subset \Sigma$ is a set for which $f(A)$, $f^2(A)$, ..., $f^{J-1}(A)$ are defined, then we denote

$$\mathcal{T}_J(A) \equiv \bigcup_{j=0}^{J-1} \bigcup_{p \in f^j(A)} \bigcup_{t \in [0,\tau(p)]} \varphi^t(p).$$

Remark 3.0.10 *For all $\epsilon > 0$ and for all $n \in \mathbb{N}$, there exists $\delta > 0$ such that if $A \subset \Sigma$ is a measurable set with $\sigma(A) < \delta$ and $f(A)$, $f^2(A)$, ..., $f^{n-1}(A)$ are defined then*

$$m(\mathcal{T}_J(A)) < \epsilon.$$

Proof: Recall that σ-a.e. point in Σ that returns to Σ belongs to $\tilde{\Sigma}_1$. By Corollary 2.1.4, there exists $\delta^* > 0$ such that $B \subset \tilde{\Sigma}_1$, $\sigma(B) < \delta^*$ implies $m(\mathcal{T}_1(A)) < \epsilon/n$.

The Poincaré map $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ is a C^1 diffeomorphism. Therefore if A_j is the subset of Σ where f^j is defined, the push-forward $f^j_*(\sigma|A_j)$ is absolutely continuous with respect to σ. So there exists $\delta > 0$ such that if $A \subset A_n$ and $\sigma(A) < \delta$ then $\sigma(f^j(A)) < \delta^*$ for all integer $0 \le j \le n-1$. Thus we conclude that $m(\mathcal{T}_1(f^j A)) < \epsilon/n$ for each such j, which yields $m(\mathcal{T}_{n-1}(A)) < \epsilon$. ∎

Let Λ be the set of points $x \in \Sigma$ such that $f^n(x)$ is well defined for all $n \in \mathbb{Z}$. Notice that this is a measurable set.

The set of points in the manifold that hit Σ infinitely many times will be denoted by M_R, that is:

$$M_R = \bigcup_{t\in\mathbb{R}} \varphi^t(\Lambda).$$

Remark 3.0.11 *The set M_R is the complement of the union of stable and unstable manifolds of the singularities of X.*

Proof: Assume that the point x is in a stable manifold of a singularity, i.e. $\varphi^t(x)$ converges to a singularity q as $t \to +\infty$. Since Σ is compact and does not contain q, the future orbit of x hits Σ at most finitely many times, showing that $x \notin M_R$.

Conversely, if a point x is in no stable or unstable manifolds of singularities then it follows from Lemma 3.0.6 that its orbit $\{\varphi^t(x)\}$ hits Σ in the future and in the past. By invariance of stable and unstable manifolds, infinitely many such hits occur. This shows that $x \in M_R$. ■

In the following remark we will show that the crushing property is already satisfied on $M\backslash M_R$, so we do not need to perturb the vector field on that set.

Remark 3.0.12 *For all $\epsilon > 0$ there exist $\bar{t} > 0$ and a compact set $K \subset M\backslash M_R$ such that*

$$m(K) > m(M\backslash M_R) - \epsilon \quad \textit{and} \quad m(\varphi^t(K)) < \epsilon \textit{ for all } t > \bar{t}.$$

Proof: Recall Remark 3.0.11. Since the stable and unstable manifolds of saddles have zero m-measure, the set $M \setminus M_R$ coincides m-mod. 0 with the union M_S of stable manifolds of sinks and unstable manifolds of saddles. We have seen in Remark 2.5.2 that this is a "self-crushing" set. ■

From now on, (f, Λ, σ) will denote the dynamical system defined by the return Poincaré map $f : \Lambda \to \Lambda$, together with the (non necessarily invariant) measure σ.

4
Tubular Chart

In this section we show that for a given C^3 vector field, it is possible to find, for a non-periodic point $p \in M$, an open neighborhood U of p and a C^2 diffeomorphism $F : F^{-1}(U) \subset \mathbb{R}^d \to U \subset M$ with some nice properties (Theorem 4.0.15). This chart will allow us to construct the perturbed vector field in the Euclidean space and to compute the volume crushing that characterizes the non-existence of acips.

Let Leb denote the Lebesgue measure on $\mathbb{R}^d$.

Definition 4.0.13 *Given a constant $C > 1$, we say that a measure $m \ll \mathrm{Leb}$, supported in some open subset $U \subset \mathbb{R}^d$, is C-*sliced *if its density*

$$\frac{dm}{d\mathrm{Leb}}(x_1, \ldots, x_d) = \omega(x_1)$$

depends only on the first coordinate and is such that

1. *ω is C^1;*
2. *$\omega(t) > 0$ for all t;*
3. *$\dfrac{\omega'(t)}{\omega(t)} \leq C$ for all t.*

Definition 4.0.14 *Let $U \subset \mathbb{R}^d$ be a Borel set. We say that two measures m_1, m_2 on U are* comparable *if*

$$\frac{1}{2} \leq \frac{m_1(S)}{m_2(S)} \leq 2,$$

for all Borel subsets $S \subset U$.

Theorem 4.0.15 (Tubular Chart's Theorem) *Given a C^3 vector field X on M and a C^2 cross-section $\Sigma \subset M$, there exists a constant $C \geq 1$ with the following properties. For any non-periodic point $p \in \Sigma$ and any $T > 0$, there exists a neighborhood V of p, an open set $U \subset \mathbb{R}^d$, and a C^2-diffeomorphism $F : U \to F(U) \subset M$ such that:*

1. $\varphi^t(V) \subset F(U)$ *for all* $t \in [-1, T+1]$;
2. $\varphi^t(p) = F(t, 0, \ldots, 0)$, *for all* $t \in [-1, T+1]$;
3. *the vector field* X *is tangent to the submanifold* $F((\mathbb{R}^{d-1} \times \{0\}) \cap U)$;
4. $F^{-1}(\Sigma \cap V) \subset \{0\} \times \mathbb{R}^{d-1}$;
5. $\mathrm{NC}\big(D(F^{-1}(q)|_{T_q\Sigma}\big) \leq C$, *for all* $q \in \Sigma \cap V$;
6. $\|DF(z)\| \leq C$, *for all* $z \in U$;
7. $\|DF(z)e_d\| \cdot \|DF^{-1}(F(z))\| \leq C$, *for all* $z \in U$ *(where* $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$*)*;
8. $\|D^2F(z)(\,\cdot\,, e_d)\| \cdot \|DF^{-1}(F(z))\| \leq C$, *for all* $z \in U$;
9. *if* m *is the Riemannian volume on* M *then* $(F^{-1})_*(m|_{F(U)})$ *is comparable to a* C*-sliced measure* $\hat{m}$ *on* U;
10. *letting* $\{P_p^t\}$ *(resp.* $\{\hat{P}_0^t\}$*) be the linear Poincaré flow with base-point* p *(resp.* 0*) for the vector field* X *on* M *(resp.* $\hat{X} \equiv (F^{-1})_*X$ *on* U*), we have* $\|P_p^{t,s}\| = \|\hat{P}_0^{t,s}\|$ *for all* $t, s \in [0, T]$.

In order to clarify the significance of this result, we comment informally how it fits in our general strategy:

- The purpose of the Theorem 4.0.15 is to put the vector field on a neighborhood of a segment of orbit in a kind of standard form in order to make it easier to find perturbations with a (local) crushing property.
- Conditions 2, 3 and 4 mean that the chart "straightens" respectively a segment of trajectory, a codimension 1 invariant submanifold containing this trajectory, and the disk Σ; see Figure 4.1.
- The change of coordinates should be uniformly controlled in several ways; this is expressed by a single control parameter C. If C were allowed to depend on the time length T, the result would be much easier; indeed, in that case one could take a change of coordinates with stronger and simpler properties. However, it is essential to our strategy that C does not depend on T.
- The diffeomorphism F can be highly non-conformal. (In fact, we will see in the proof that the expansion rates along hyperplanes $\{t\} \times \mathbb{R}^{d-1}$ can be much smaller than along the line $\mathbb{R} \times \{0\}^{d-1}$.) Nevertheless, its restriction to $\Sigma \cap V$ is approximately conformal, as stated in condition 5.

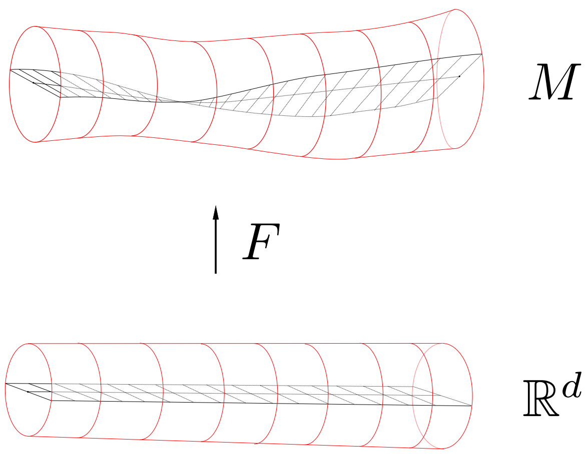

Figure 4.1: Tubular Chart

- We follow the strategy of [AB1] and try to crush volumes in one dimension only, and so to make d-dimensional objects essentially $(d-1)$-dimensional. We will crush volume towards the codimension 1 submanifold $F((\mathbb{R}^{d-1} \times \{0\}) \cap U)$ Under the change of coordinates provided by the theorem, the new vector field needs only to be perturbed along the direction of the d-th coordinate. We call such perturbations *vertical.*
- By pulling back a vertical perturbation of $\hat{X} = (F^{-1})_* X$, we should obtain a C^1-perturbation of X. Clearly, an upper bound on the C^1-distance of pulled-back vector fields should depend on the derivatives of F and F^{-1} up to second order. As we will see later, technical conditions 6, 7 and 8 are precisely what is needed to make such control possible for vertical perturbations.
- It would be nice if the map F^{-1} sent Riemannian volume in M to a Lebesgue measure in $\mathbb{R}^d$ (or a constant multiple of it); however it seems difficult to impose this extra requirement. We notice, however, that to study the crushing property we can replace a measure by a comparable one (in the sense of Definition 4.0.14.) Condition 9 in Theorem 4.0.15 means that F^{-1} sends Riemannian volume in M to something comparable to Lebesgue measure in $\mathbb{R}^d$ times a factor which varies slowly with respect to time (in the sense of the last condition in Definition 4.0.13). Those conditions will be sufficient for our strategy to work, because our crushing estimates are basically done in “time snapshots” (similarly to what happens in [AB1]).
- Condition 10 implies the norm of Poincaré flow for the new vector field $\hat{X}$ grows at most as much as fast as for X. This technical condition is needed for the construction of the crushing perturbations.

– The construction of the chart F uses the orthonormal frame flow (see § 2.4), whose class of differentiability is one less than the flow on M. Since we need F to be C^2, we ask X to be C^3. And it is of course necessary to ask Σ to be C^2, in view of condition 4.

After those remarks, let us now prove Theorem 4.0.15:

Proof: By the Whitney embedding theorem, we can assume that M is embedded in $\mathbb{R}^N$, for some large $N > d$. Moreover, by the Nash Embedding Theorem, we can assume that the Riemannian metric on M is inherited from the Euclidean metric on $\mathbb{R}^N$. (One could avoid appealing to Nash's theorem by noticing that, since M is compact, a change of Riemannian metric is absorbed by a change of the constants in the statement of Theorem 4.0.15. Alternatively, since our main theorem does not depend on the choice of the Riemannian metric, we could have fixed a priori any suitable Riemannian metric to work with.)

Fix a normal tubular neighborhood $M^\epsilon \subset \mathbb{R}^N$ of M of some width $\epsilon > 0$, and the associate bundle projection $\pi : M^\epsilon \to M$; more precisely, $M^\epsilon = \{z \in \mathbb{R}^N : d(z, M) \leq \epsilon\}$, and for each $z \in M^\epsilon$, $\pi(z)$ is the point in M which is closest to z.

Fix $X \in \mathfrak{X}^3(M)$. For any point $p \in M$ and any orthonormal frame $\mathfrak{f} = (v_1, \ldots, v_d)$ at T_pM, we will define a map $G_{p,\mathfrak{f}} : \mathbb{R} \times B_\epsilon \to M$, where B_ϵ is the closed ball in $\mathbb{R}^{d-1}$ of center 0 and radius ϵ, as follows. Let $\{(v_1(t), \ldots, v_d(t))\}_{t \in \mathbb{R}}$ be the trajectory of the orthonormal frame flow induced by X (recall § 2.4), with initial conditions

$$v_i(0) = v_i, \quad 1 \leq i \leq d.$$

Then we define

$$\begin{array}{cccl} G_{p,\mathfrak{f}}\colon & \mathbb{R} \times B_\epsilon & \to & M \\ & (x_1, x_2, \ldots, x_d) & \mapsto & \pi\left(\varphi^{x_1}(p) + \sum_{j=2}^d x_j v_j(x_1)\right) \end{array}$$

Since the orthonormal frame flow is C^2 (because X is C^3), this map is C^2. Moreover, by compactness of the orthonormal frame bundle, we can find a constant C_0 such that

$$\|DG_{p,\mathfrak{f}}(z)\| \leq C_0, \qquad \|D^2 G_{p,\mathfrak{f}}(z)\| \leq C_0, \tag{4.1}$$

for all $p \in \Sigma$, all orthonormal frames $\mathfrak{f} \in \mathfrak{F}_p$, and all $z \in \mathbb{R} \times B_\epsilon$.

Now assume that $p \in M$ is nonsingular (i.e., $X(p) \neq 0$) and $\mathfrak{f} \in \mathfrak{F}_p$ satisfies

$$\mathfrak{f} = (v_1, \ldots, v_d) \quad \text{where} \quad v_1 = \frac{X(p)}{\|X(p)\|}\,. \tag{4.2}$$

Since $G(x_1, 0, \ldots, 0) = \varphi^{x_1}(p) \in M$, and π is a C^∞ retraction onto M, the partial derivatives of $G_{p,\mathfrak{f}}$ at $(x_1, 0 \ldots, 0)$ are given by:

$$DG_{p,\mathfrak{f}}(x_1, 0, \ldots, 0) \cdot e_j = \begin{cases} X(\varphi^{x_1}(p)) = \|X(\varphi^{x_1}(p))\| v_1(x_1) & \text{if } j = 1, \\ v_j(x_1) & \text{if } j \geq 2, \end{cases} \tag{4.3}$$

where $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{R}^d$. In particular, the map $G_{p,\mathfrak{f}}$ is a local diffeomorphism at each point in the line $\mathbb{R} \times \{0\}^{d-1}$ (under the assumptions $X(p) \neq 0$ and (4.2)).

Next, fix a C^2 cross-section $\Sigma \subset M$. Notice that the pairs $(p, \mathfrak{f})$ where $p \in \Sigma$ and $\mathfrak{f} \in \mathfrak{F}_p$ satisfies (4.2) form a compact set. Since Σ is C^2 and transverse to X, for each such p and $\mathfrak{f}$, there is a neighborhood V_p of p such that

$$G_{p,\mathfrak{f}}^{-1}(\Sigma \cap V_p) = \big\{(x, u) \in \mathbb{R} \times \mathbb{R}^{d-1} : x = g_{p,\mathfrak{f}}(u)\big\}, \tag{4.4}$$

where $g_{p,\mathfrak{f}}$ is a C^2 function on a open neighborhood of 0 in $\mathbb{R}^{d-1}$. By compactness, there is a constant C_1 such that

$$\|Dg_{p,\mathfrak{f}}(0)\| \leq C_1, \qquad \|D^2 g_{p,\mathfrak{f}}(0)\| \leq C_1, \tag{4.5}$$

for all $p \in \Sigma$ and $\mathfrak{f}$ satisfying (4.2). Also notice that $g_{p,\mathfrak{f}}(0) = 0$.

Now fix $p \in \Sigma$ and $T > 0$. The constant C that appears in the statement of the Theorem will be exhibited later, but it will not depend on p and T.

Let v_1 be given by (4.2). Choose some unit vector

$$v_d \in T_p\Sigma \cap (X(p))^\perp \tag{4.6}$$

(which is possible because we are assuming that $d \geq 3$). Next, choose vectors $v_2, \ldots, v_{d-1}$ such that $\mathfrak{f} = (v_1, v_2, \ldots, v_d)$ is an orthonormal d-frame on T_pM. (See Figure 4.2.)

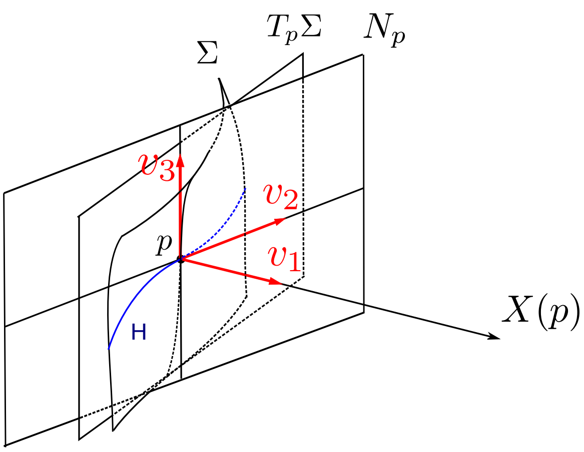

Figure 4.2: Choice of the initial orthonormal frame for $d = 3$.

Define

$$\alpha \equiv \min_{t\in[-2,T+2]} \|X(\varphi^t(p))\| \,. \tag{4.7}$$

For simplicity of notation, let $G = G_{p,\mathfrak{f}}$ and $g = g_{p,\mathfrak{f}}$. Define the following linear isomorphism

$$\begin{array}{rccl} L_\alpha\colon & \mathbb{R}^d & \to & \mathbb{R}^d \\ & (x_1, x_2, \ldots, x_d) & \mapsto & (x_1, \alpha x_2, \ldots, \alpha x_d) \end{array}$$

Let

$$F_1 = G \circ L_\alpha \,.$$

So (4.3) gives

$$DF_1(x_1, 0, \ldots, 0) = \begin{pmatrix} \|X(\varphi^{x_1}(p))\| & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix}, \tag{4.8}$$

where the matrix is relative to the bases $(e_1, \ldots, e_d)$ in $\mathbb{R}^d$ and $(v_1(x_1), \ldots, v_d(x_1))$ in $T_{\varphi^{x_1}(p)}M$. By the inverse function theorem, there exists a neighborhood U_1 of $[-2, T+2] \times \{0\}^{d-1}$ such that $F_1|U_1$ is a diffeomorphism onto an open subset of M.

Notice that F_1 already satisfies property 2, that is, $F_1(t, 0, \ldots, 0) = \varphi^t(p)$. The role of F_4 is basically to straighten two codimension 1 submanifolds in order to obtain properties 3 and 4.

We split $\mathbb{R}^d$ as $\mathbb{R} \times \mathbb{R}^{d-2} \times \mathbb{R}$ and take coordinates (x, w, y) with $x \in \mathbb{R}$, $w \in \mathbb{R}^{d-2}$, $y \in \mathbb{R}$.

If follows from (4.4) that for a sufficiently small neighborhood $V \ni p$,

$$F_1^{-1}(\Sigma \cap V) = \big\{(x, w, y) \in \mathbb{R} \times \mathbb{R}^{d-1} : x = g(\alpha w, \alpha y)\big\}. \tag{4.9}$$

Recalling the choice (4.7) of v_d, we obtain:

$$\frac{\partial g}{\partial y}(0, 0) = Dg(0, 0) \cdot e_d = 0 \,. \tag{4.10}$$

Define a diffeomorphism on a neighborhood of $[-2, T+2] \times \mathrm{dom}(g) \subset \mathbb{R}^d$ by

$$F_2(x, w, y) = \big(x - g(\alpha w, \alpha y),\ w,\ y\big).$$

So $F_2 \circ F_1^{-1}(\Sigma \cap V) \subset \{0\} \times \mathbb{R}^{d-2} \times \mathbb{R}$. Let $\{\tilde{\varphi}^t\}$ be the flow of the vector field $(F_2 \circ F_1^{-1})_* X$. Let H be a small neighborhood of 0 in $\{0\} \times \mathbb{R}^{d-2} \times \{0\}$. Then

$$\tilde{H} \equiv \bigcup_{t\in[-1,T+1]} \tilde{\varphi}^t(H)$$

is a codimension 1 submanifold of $\mathbb{R}^d$ containing the line $[-1, T+1] \times \{0\}^{d-2} \times \{0\}$.

Claim 4.0.16 *The tangent space of $\tilde{H}$ at any point of this line is $\mathbb{R} \times \mathbb{R}^{d-2} \times \{0\}$.*

Proof of the Claim: It follows from the definition of the orthonormal frame flow that

$$D\varphi^t(p) \cdot \operatorname{span}(v_1, \dots, v_{d-1}) = \operatorname{span}(v_1(t), \dots, v_{d-1}(t)).$$

Notice that the image of this space under $D(F_2 \circ F_1^{-1})(\varphi^t(p))$ is exactly the tangent space of $\tilde{H}$ at $(t, 0, 0)$. The claim follows. ∎

It follows from the claim that, reducing H if necessary, the manifold $\tilde{H}$ is the graph of a function:

$$\tilde{H} = \{(x, w, y) : y = h(x, w)\},$$

where $h : \operatorname{dom}(h) \subset \mathbb{R} \times \mathbb{R}^{d-2} \to \mathbb{R}$ satisfies

$$h(x, 0) = 0 \quad \text{and} \quad Dh(x, 0) = 0\,. \tag{4.11}$$

(See Figure 4.3.)

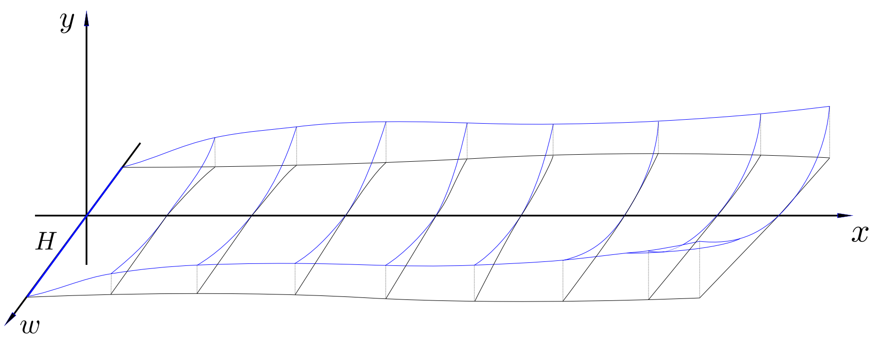

Figure 4.3: The manifold $\tilde{H}$ as a graph

Define a diffeomorphism

$$F_3(x, w, y) = \big(x,\ w,\ y - h(x, w)\big).$$

So $F_3(\tilde{H}) \subset \mathbb{R} \times \mathbb{R}^{d-1} \times \{0\}$. The compose map

$$F_3 \circ F_2(t, w, y) = \big(x - g(\alpha w, \alpha y),\ w,\ y - h(x - g(\alpha w, \alpha y), w)\big)$$

is a diffeomorphism; let $F_2 = (F_3 \circ F_2)^{-1}$, i.e.,

$$F_4(x, w, y) = \big(x + g(\alpha w, \alpha y + \alpha h(x, w)),\ w,\ y + h(x, w)\big)\,.$$

Let us check that $F = F_1 \circ F_4$ satisfies all properties in the statement of the Theorem.

We have already mentioned that F_1 satisfies property 2. Since F_4 fixes $\mathbb{R} \times \{0\}^{d-1}$, the map F will clearly inherit this property.

Properties 4 and 3 are straightforward.

It follows from Property 4, that

$$DF^{-1}(q)|_{T_q\Sigma} = (DF(F^{-1}(q)))^{-1}|_{\{0\}\times\mathbb{R}^{d-1}}. \tag{4.12}$$

Thus, in order to check Property 5, observe that $DF(0) \cdot e_j = \alpha v_j$ for $j = 2, \dots, d$. In particular, $DF(0)$ is conformal. Taking $C \geq 2$, Property 5 follows by taking a sufficiently small neighborhood $\tilde{V} = F^{-1}(V)$ of zero.

Using (4.11) and (4.10), we see that the derivative of F_4 on the points in $\mathbb{R} \times \{0\}^{d-2} \times \{0\}$ has the following (block) matrix expression:

$$DF_4(t, 0, 0) = \begin{pmatrix} 1 & \alpha \frac{\partial g}{\partial w}(0,0) & 0 \\ 0 & \mathrm{id}_{d-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.13}$$

In particular, using (4.5) and the fact that $\alpha \leq \|X\|_{C^0}$, we obtain

$$\|(DF_4(x, 0, 0))^{\pm 1}\| \leq C_2, \tag{4.14}$$

where C_2 depends only on X and Σ. Thus, reducing U if necessary, we can assume that

$$\|(DF_4(z))^{\pm 1}\| \leq 2C_2 \quad \text{for all } z \in U. \tag{4.15}$$

It follows from (4.7) and (4.8) that $\|DF_1(x, 0, 0)\| \leq \|X\|_{C^0}$. Reducing U if necessary, we can assume that

$$\|DF_1\| \leq 2\|X\|_{C^0} \quad \text{on } F_4(U). \tag{4.16}$$

Since $F = F_1 \circ F_4$, it follows from (4.15) and (4.16) that property 6 is satisfied, provided the constant C is chosen bigger that $4C_2\|X\|_{C^0}$.

Using (4.8) and (4.13), we have

$$DF(x, 0, 0)e_d = DF_1(F_4(x, 0, 0)) \cdot DF_4(x, 0, 0)e_d$$

$$= DF_1(x, 0, 0)e_d = \alpha v_d(x).$$

Reducing U, we obtain

$$\|DF(z)e_d\| \leq 2\alpha \quad \text{for all } z \in U\,. \tag{4.17}$$

It follows from (4.7) and (4.8) that $\|DF_1^{-1}(\varphi^x(p))\| = \alpha^{-1}$. So, using (4.14), we have

$$\|DF^{-1}(\varphi^x(p))\| \leq C_2\alpha^{-1},$$

for all $x \in [0, T]$. Reducing U, if necessary, we obtain

$$\|DF^{-1}(F(z))\| \leq 2C_2\alpha^{-1}, \quad \text{for all } z \in U. \tag{4.18}$$

Putting this together with (4.17), we obtain

$$\|DF(z)e_d\| \cdot \|DF^{-1}(F(z))\| \leq 4C_2\,;$$

that is, property 7 is verified, provided we choose $C \geq 4C_2$.

Let us check property 8. First observe that the linear map $D^2F(z)(e_d, \cdot)$ is the derivative of the map

$$\begin{aligned} z = (x, w, y) \mapsto\ & DF(z) \cdot e_d \\ &= DG(L_\alpha \circ F_4(z)) \circ L_\alpha \circ DF_4(z) \cdot e_d \\ &= DG(L_\alpha \circ F_4(z)) \cdot \left(\alpha \frac{\partial g}{\partial y}(w, y) \cdot e_1 + \alpha \cdot e_d \right) \\ &= \alpha\Psi(z), \end{aligned}$$

where we define Ψ as

$$\Psi(x, w, y) = DG(L_\alpha \circ F_4(z)) \cdot \left(\frac{\partial g}{\partial y}(w, y) \cdot e_1 + e_d \right)$$

Using (4.1), (4.15), (4.5), and that $\alpha \leq \|X\|_{C_0}$, we see that $\|D\Psi\| \leq C_3$, for some constant C_3 depending only on X and Σ. That is, $\|D^2F(z)(e_d, \cdot)\| \leq C_3\alpha$. Putting this together with (4.18), we conclude that property 8 is satisfied, provided $C \geq 2C_2C_3$.

Let us check Property 9. For that matter, consider the measure $\hat{m}$ defined by

$$\hat{m}(S) = \int_S \alpha^{d-1}\|X(\varphi^t(p))\| dt dx_2 \ldots dx_d,$$

where $S \subset U$ is a Borel set in $\mathbb{R}^d$.

Notice that we can represent DF_1 as a matrix that sends the orthonormal base $\{e_1, e_d, \ldots, e_d\}$ of $\mathbb{R}^d$ to the orthonormal base $\{v_1(t), v_2(t), \ldots, v_d(t)\}$ of $T_{\varphi^t(p)}M$. Thus the Jacobian of F_1 is the determinant of such matrix. Using

(4.8) and (4.13), we see that the Jacobian of F along $(t, 0, \ldots, 0)$ is

$$\mathrm{Jac}(F)(t, 0, \ldots, 0) = \alpha^{d-1}\|X(\varphi^t(p))\|.$$

Therefore, we can reduce U if necessary, to obtain

$$\frac{1}{2} \leq \frac{\mathrm{Jac}(F)(z)}{\alpha^{d-1}\|X(\varphi^t(p))\|} \leq 2, \tag{4.19}$$

for all $z \in U$.

By the change of variables formula,

$$F_*^{-1}(m)(S) = m(F(S)) = \int_S \mathrm{Jac}(F)(t, x_1, \ldots, x_d)dtdx_2 \ldots dx_d,$$

which together with (4.19) leads us to conclude that $\hat{m}$ is comparable to $F_*^{-1}(m)|_{F(U)}$. In order to show that $\hat{m}$ is a C-sliced measure, observe that if $\omega(t) = \alpha^{d-1}\|X(\varphi^t(p))\|$, then

$$\begin{aligned}
\omega'(t) &\leq \alpha^{d-1}\left\|\frac{dX(\varphi^t(p))}{dt}\right\| \\
&\leq \alpha^{d-1}\|DX(\varphi^t(p)) \cdot X(\varphi^t(p))\| \\
&\leq \alpha^{d-1}\|DX\|_{C^0} \cdot \|X(\varphi^t(p))\| \\
&\leq C\omega(t),
\end{aligned}$$

provided that the constant C is chosen bigger then the C^1-norm of X.

It only remains to check Property 10. For that end, consider the canonical basis in $\mathbb{R}^d$ and the basis $(v_1(t), \ldots, v_d(t))$ at the tangent space of M at $\varphi^t(p)$. We can express linear maps as matrices according to those bases. Thus:

$$\begin{aligned}
D\hat{\varphi}^s(t, 0, 0) &= (DF(t+s, 0, 0))^{-1} \circ D\varphi^s(\varphi^t p) \circ DF(t, 0, 0) \\
&= \begin{pmatrix} \|X(\varphi^{t+s}p)\|^{-1} & * \\ 0 & \alpha^{-1}\mathrm{id} \end{pmatrix} \begin{pmatrix} \frac{\|X(\varphi^{t+s}p)\|}{\|X(\varphi^t p)\|} & * \\ 0 & P_p^{t,s} \end{pmatrix} \begin{pmatrix} \|X(\varphi^t p)\| & * \\ 0 & \alpha\mathrm{id} \end{pmatrix} \\
&= \begin{pmatrix} 1 & * \\ 0 & P_p^{t,s} \end{pmatrix}.
\end{aligned}$$

So the matrices of $\hat{P}_0^{t,s}$ and $P_p^{t,s}$ coincide. Since we are taking matrices with respect to orthonormal bases, Property 10 is satisfied. ■

Remark 4.0.17 *Notice that Theorem 4.0.15 provides no uniform estimate for the C^1 norm of the new vector field $\hat{X} = (F^{-1})_* X$. It neither provides an* *estimate for the C^2 norm of F (and in fact, $\|F\|_{C^2}$ can be arbitrarily large, as shown by Example 4.0.18 below). However, no such estimates will be necessary.*

Example 4.0.18 *Let us exhibit one example where $\|F\|_{C^2}$ can be arbitrarily large. The example will be constructed in $M = \mathbb{R}^3$, but it is easy to adapt the construction to a compact manifold M of dimension $d = 3$. For $(x, w, y) \in \mathbb{R}^3$, define $X(x, w, y) = (1, 0, w^2)$. The flow induced by X is given by*

$$\varphi^t(x_0, w_0, y_0) = (x_0 + t, w_0, y_0 + w_0^2 t).$$

If $p = (0, 0, 0)$, Property 2 is already satisfied and, in particular, for any $T > 0$ we have $\alpha = 1$. Suppose Σ is a disc in $\mathbb{R} \times \{0\} \times \mathbb{R}$. By (4.2) we have $v_1 = (1, 0, 0)$; suppose we choose $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$. Then the frame $(v_1(t), v_2(t), v_3(t))$ does not depend on t and $\tilde{H}$ is the graph of $h(x, w) = xw^2$ (See Figure 4.4). Since we are already placed in $\mathbb{R}^3$ and in a context where the cross-section Σ and the base orbit are already “straight”, the role of the diffeomorphism F is to straighten $\tilde{H}$, that is

$$F(x, w, y) = F_4(x, w, y) = (x, w, y + h(x, w)).$$

Observe that the curvature of the surface $\tilde{H}$ along the x-axis tends to infinity. In fact,

$$\|D^2F(x, 0, 0)\| \geq \left| \frac{\partial^2 h(x, 0)}{\partial w^2} \right| = 2|x|.$$

Therefore, the second derivative of F is unbounded.

Let $F : U \subset \mathbb{R}^d \to F(U) \subset M$ be given by Theorem 4.0.15. As explained above, we need to compare the C^1 norm of a vector field $\hat{Y} \in \mathfrak{X}^1(U)$ and its push-forward $F_*\hat{Y} \in \mathfrak{X}^1(F(U))$. Actually we will only study this problem for *vertical* vector fields $\hat{Y}$; the norm comparison is then given by the following:

Proposition 4.0.19 *Let $X \in \mathfrak{X}^3(M)$ and $F : U \subset \mathbb{R}^d \to F(U) \subset M$ be given by Theorem 4.0.15. If $\Psi : \mathbb{R}^d \to \mathbb{R}$ is a C^1 map and $\hat{Y} \in \mathfrak{X}^1(\hat{U})$ is a vector field of the form*

$$(x_1, x_2, \ldots, x_d) \to (0, \Psi(x_1, x_2, \ldots, x_d), 0, \ldots, 0),$$

then

$$\|F_*\hat{Y}\|_{C^1} \leq 2C\|\hat{Y}\|_{C^1},$$

where $C > 1$ is the constant given by Theorem 4.0.15.

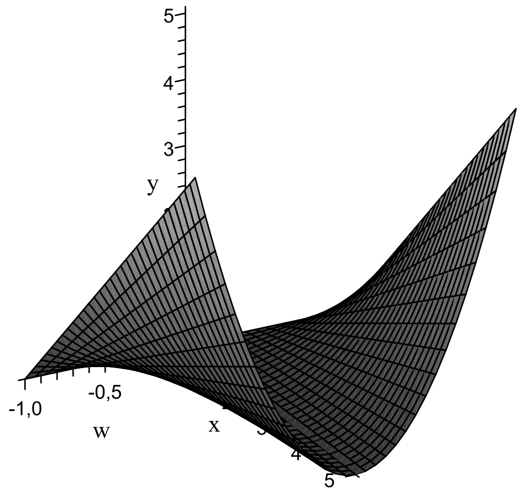

Figure 4.4: The 1-codimensional submanifold $\tilde{H} = \{(x, w, y) : y = xw^2\}$ in Example 4.0.18 is graph of a function with unbounded second derivative.

Proof: Let us denote $Y = F_* \hat{Y}$. First, note that

$$\|Y\|_{C^0} \leq \max_{z \in U} \|DF(z)\| \cdot \|\hat{Y}\|_{C^0}.$$

From property 6 in Theorem 4.0.15, we obtain

$$\|Y\|_{C^0} \leq C\|\hat{Y}\|_{C^0}.$$

Now, let us estimate the norm of the derivative. Observe that for a given $z \in U$ and v in TT_zM (which we can identify with T_zM, since M is embedded in some $\mathbb{R}^N$), we have

$$DY(z) \cdot v = A \cdot v + B \cdot v,$$

where

$$A \cdot v = DF(F^{-1}(z)) \cdot D\hat{Y}(F^{-1}(z)) \cdot DF^{-1}(z) \cdot v$$

and

$$B \cdot v = D^2F(F^{-1}(z))(DF^{-1}(z) \cdot v, \hat{Y}(F^{-1}(z))).$$

In order to estimate $\|A\|$, note that $D\hat{Y}(p) \cdot w = (D\Psi(p) \cdot w) \cdot e_d$ and $\|D\Psi(p)\| \leq \|\hat{Y}\|_{C^1}$. Then

$$\begin{aligned} \|A\| &\leq \|DF(F^{-1}(z)) \cdot D\hat{Y}(F^{-1}(z))\| \cdot \|DF^{-1}(z)\| \\ &\leq \|DF(F^{-1}(z)) \cdot e_d\| \cdot \|\hat{Y}\|_{C^1} \cdot \|DF^{-1}(z)\|. \end{aligned}$$

From property 7 in Theorem 4.0.15,

$$\|A\| \leq C\|\hat{Y}\|_{C^1}.$$

In order to estimate $\|B\|$, note that $\hat{Y}(F^{-1}(z)) = \Psi(F^{-1}(z)) \cdot e_d$ and $\|\Psi(F^{-1}(z))\| \leq \|\hat{Y}\|_{C^1}$. Then

$$\|B\| \leq \|D^2F(F^{-1}(z)(e_d, \cdot)\| \cdot \|\hat{Y}\|_{C^1} \cdot \|DF^{-1}(z)\|.$$

From property 8 of Theorem 4.0.15 we obtain

$$\|B\| \leq C\|\hat{Y}\|_{C^1}$$

and conclude that

$$\|DY(z)\| \leq 2C\|\hat{Y}\|_{C^1},$$

as claimed. ■

Now that we have presented the type of tubular chart we need in the proof of our result, we can define a κ-rectangle - a set with dimension $(d-1)$, transverse to the flow and with a specific geometry that will meet our future needs.

Definition 4.0.20 *Given $0 < \kappa < 1$ we say that $U_0 \subset \Sigma$ is a κ-rectangle if there exists $\rho > 0$ and a tubular chart $F : U \to M$ such that*

$$F(\{0\} \times [-\kappa\rho, \kappa\rho] \times [-\rho, \rho]^{d-2}) = U_0.$$

Remark 4.0.21 *The bounded eccentricity of the Euclidean κ-rectangles implies clearly that they form a Vitali Cover of $\{0\} \times \mathbb{R}^{d-1}$. By Item (5) of Theorem 4.0.15 and Remark 2.7.5, we conclude that κ-rectangles form a Vitali Cover of $\Sigma \cap V$.*

5
Local Crushing

In this section we show how to perturb the vector field inside a tubular neighborhood of an orbit segment in order to obtain the crushing property defined in Lemma 2.5.1, relatively to the volume of the neighborhood. The name "Fettuccine" given for the main lemma of the section is due to the fact that crushable sets must have a specific geometry which resembles the fettuccine's shape. We now state this lemma (Lemma 5.0.22), which will be proved only in §5.4, after we present in the following subsections the necessary ingredients of the proof.

Lemma 5.0.22 (Fettuccine's Lemma) *Let $X \in \mathfrak{X}^3(M)$ and let Σ be a cross section. Then for all $\epsilon > 0$ and $0 < \delta < 1$ there exists $t_0 > 0$ such that for all $T_0 > 3t_0$ there exists $\kappa > 0$ such that for all $T \in (3t_0, T_0)$ and for all non-periodic point $p \in \Sigma$, there exists $\rho > 0$ such that given a κ-rectangle R centered in p with diam$(R) < \rho$, there exists $\tilde{X} \in \mathfrak{X}^1(M)$ with $\|\tilde{X} - X\|_{C^1} < \epsilon$, $\tilde{X} = X$ outside U, where*

$$U = \bigcup_{t \in [0,T]} \varphi^t(R),$$

and there exists $V \subset U$ such that if

$$U^- = \bigcup_{t \in [0,T-t_0]} \varphi^t(R) \quad \text{and} \quad U^+ = \bigcup_{t \in [t_0,T]} \varphi^t(R),$$

then

1. *$V \subset U^-$;*
2. *$\dfrac{m(V)}{m(U^-)} > 1 - \delta$;*
3. *$\varphi^t_{\tilde{X}}(\overline{V}) \subset U \qquad \forall t \in [0, t_0]$;*
4. *$\dfrac{m(\varphi^{t_0}_{\tilde{X}}(V) \triangle U^+)}{m(U^+)} < \delta$.*

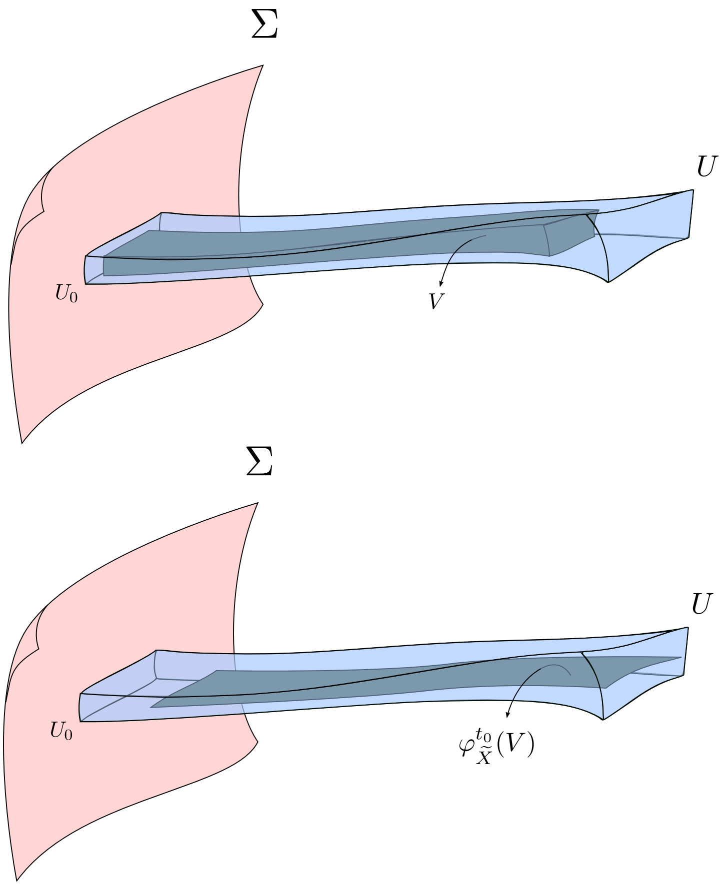

Figure 5.1: Schematic illustration of V being crushed.

The Tubular Chart Theorem (Theorem 4.0.15) set us in a very useful geometrical structure. Pulling back a vector field by the tubular chart, we obtain a vector field in an open set of $\mathbb{R}^d$ with several properties that will be frequently used in next subsections. For sake of clarity we call a vector field with those properties a *model* vector field.

Definition 5.0.23 (Model Vector Field) *Let $a, C > 0$ and $T > 2a$ be arbitrary constants. Let $X \in \mathfrak{X}^1(U)$ be a vector field defined in an open neighborhood U of the line $\{(t,0,0) \in \mathbb{R}\times\mathbb{R}^{d-2}\times\mathbb{R} : t \in [-a, T+a]\}$. We say that X is a* (C,T,a) - model vector field *if*

- $X(s,0,0) = (1,0,0), \qquad \forall s \in [-a, T+a],$
- *the plane $\{(x,w,y) : y = 0\}$ is X-invariant,*
-
$$\left|\frac{d}{dt}\log\|P_s^t\|\right| \leq C, \quad \forall s,t \in [-a, T+a],$$
where P_s^t is the linear isomorphism induced by the linear Poincaré flow based on the segment of orbit $\{(x,0,0) : x \in [s,t]\}$.

Depending on the context, some of the constants C, T and a may be omitted from the notation of model vector fields. Possibly for they are implicitly understood, possibly for being unnecessary in some computation.

5.1 Crushing-Time

Once we have constructed a chart which permits us to work with model vector fields, next step will be defining the crushing-perturbation of a general model vector field. The main idea is to add small vertical vectors to the original vector field, bringing trajectories closer to the invariant plane and, consequently, crushing volume in that direction (See Figure 5.2). Since these new vertical components need to be very small, in order to obtain a significant "vertical deviation" of the original trajectories, a long period of time must elapse. This amount of time will be called *crushing-time* and its precise definition is given by Lemma 5.1.2.

Let $\pi_d : \mathbb{R} \times \mathbb{R}^{d-2} \times \mathbb{R}$ the standard projection in the d-th coordinate.

Then $D\pi_d(x, w, y)X(x, w, y)$ is simply the d-th coordinate of the vector $X(x, w, y)$. Define $\alpha : \mathbb{R} \times \mathbb{R}^{d-2}$ by

$$\alpha(x, w) = D(\pi_d \circ X)(x, w, 0) \cdot e_d.$$

Since X is C^1 we can write

$$D\pi_d(x, w, y)X(x, w, y) = \alpha(x, w)y + r(x, w, y), \tag{5.1}$$

where $r : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a continuous function satisfying

$$\lim_{y \to 0} \frac{r(x, w, y)}{y} = 0. \tag{5.2}$$

Proposition 5.1.1 *For all $\epsilon' > 0$ and $T_2 > T_1 > 0$ there exist $\rho > 0$ and such that if* $\max\{\|w\|, |y|\} < \rho$ *and* $x \in [0, T_1]$ *then*

$$\|\varphi^t(x, w, y) - (x + t, 0, 0)\| < \epsilon',$$

for all $t \in [0, T_2 - T_1]$, where φ^t is the flow generated by the vector field X.

Proof: This is an immediate consequence of the continuity of the flow and the first property of model vector fields. ∎

Lemma 5.1.2 *Let $\epsilon > 0$ and $0 < \delta < 1$ be given. Then for any*

$$T > t_0 \equiv 3 - \frac{2\log(\delta)}{\epsilon},$$

for all $0 < a < 1$ *and for all model vector field* $X \in \mathfrak{X}^1(U)$ *(with respect to* a *and* T*), there exists* $\rho > 0$ *such that if* $z = (x, w, y) \in U$ *satisfies*

$$\max\{\|w\|, |y|\} < \rho \quad \textit{and} \quad x \in [0, T - t_0]$$

then

$$\widetilde{X}(x, w, y) = X(x, w, y) + (0, 0, -\epsilon y)$$

would be such that

$$\frac{|\pi_d(\varphi_{\widetilde{X}}^{\tilde{\tau}(z)}(z))|}{|\pi_d(\varphi_X^{\tau(z)}(z))|} < \delta,$$

where $\tilde{\tau}, \tau : N_{(x,0,0)} \to N_{(x+t_0,0,0)}$ *are the hitting-time functions defined in Subsection (2.1) with respect to time* t_0 *and to the fields* $\widetilde{X}$ *and* X*, respectively.*

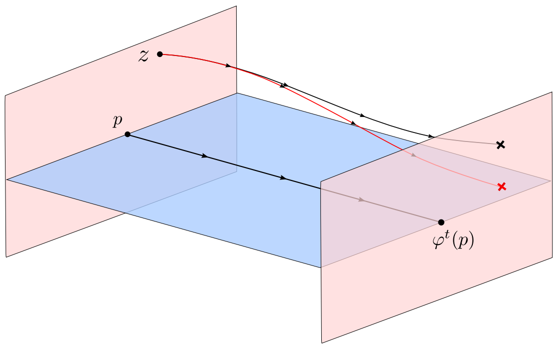

Figure 5.2: Crushing in the y-direction.

Proof: Notice that $\widetilde{X}$ is also a model vector field and thus we can obtain an equation similar to (5.1). Namely

$$D\pi_d(x, w, y) \cdot \widetilde{X}(x, w, y) = (\alpha(x, w) - \epsilon)y + r(x, w, y). \tag{5.3}$$

Suppose we have fixed a point $z = (x, w, y) \in \mathbb{R}^d$ sufficiently close to the x-axis (we will see later what this means). Let $\varphi_X^t(z) = (x(t), w(t), y(t))$ and $\varphi_{\widetilde{X}}^t(z) = (x_\epsilon(t), w_\epsilon(t), y_\epsilon(t))$ be the future paths of z generated by X and $\widetilde{X}$, respectively.

Then we have

$$\begin{aligned} y'_\epsilon(t) &= [\pi_d(\varphi_{\widetilde{X}}^t(z))]' \\ &= D\pi_d(\varphi_{\widetilde{X}}^t(z))\widetilde{X}(\varphi_{\widetilde{X}}^t(z)) \\ &= (\alpha(x_\epsilon(t), w_\epsilon(t)) - \epsilon)y_\epsilon(t) + r(x_\epsilon(t), w_\epsilon(t), y_\epsilon(t)). \end{aligned} \tag{5.4}$$

Similar computations lead us to

$$y'(t) = \alpha(x(t), w(t))y(t) + r(x(t), w(t), y(t)). \tag{5.5}$$

Proposition (5.1.1) together with the continuity of function α permit us to take $\rho > 0$ sufficiently small such that if $\max\{\|w\|, |y|\} < \rho$ and $x \in [-a, T - t_0]$ then

$$|\alpha(x(t), w(t)) - a_x(t)| < \epsilon/8 \quad \text{and} \quad |\alpha(x_\epsilon(t), w_\epsilon(t)) - a_x(t)| < \epsilon/8,$$

for all $t \in [0, t_0]$, where $a_x(s) \equiv \alpha(x + s, 0)$. Notice that the choice of ρ depends on T but does not depend on x.

By equation (5.2) and by the compactness of V, we can reduce ρ, if necessary, to guarantee that

$$|\tilde{r}(x_\epsilon(t), w_\epsilon(t), y_\epsilon(t))| \leq \frac{\epsilon|y_\epsilon(t)|}{8} \quad \text{and} \quad |r(x(t), w(t), y(t))| \leq \frac{\epsilon|y(t)|}{8},$$

for all $t \in [0, t_0]$ and for all (x, w) in the compact domain.

From these estimates and from equations (5.4) and (5.5) we can deduce the following inequalities:

$$y'_\epsilon(t) \leq (a_x(t) - 7\epsilon/8)y_\epsilon(t) + r(x_\epsilon(t), w_\epsilon(t), y_\epsilon(t)) \leq (a_x(t) - 3\epsilon/4)y_\epsilon(t) \tag{5.6}$$

$$y'(t) \geq (a_x(t) - \epsilon/8)y(t) + r(x(t), w(t), y(t)) \geq (a_x(t) - \epsilon/4)y(t). \tag{5.7}$$

Besides that, we want to consider $\rho > 0$ small enough (as in Remark 2.1.6) to obtain

$$|\tau(z) - t_0| < \min\left\{\frac{a}{2}, \frac{\epsilon}{4M}\right\} \quad \text{and} \quad |\tilde{\tau}(z) - t_0| < \min\left\{\frac{a}{2}, \frac{\epsilon}{4M}\right\},$$

where

$$M = \sup_{s \in [-a, T+a]} |\alpha(s, 0)|.$$

Once we have all these estimates, we can conclude that

$$\frac{y_\epsilon(\tilde{\tau}(z))}{y(\tau(z))} \leq \frac{y_\epsilon(0)\exp(-3\epsilon\tilde{\tau}(z)/4)\exp\left(\int_0^{\tilde{\tau}(z)} a_x(t)dt\right)}{y(0)\exp(-\epsilon\tau(z)/4)\exp\left(\int_0^{\tau(z)} a_x(t)dt\right)}$$

$$\leq \exp(-3\epsilon(t_0-1)/4+\epsilon(t_0+1)/4)\exp\left(|\tilde{\tau}(z)-\tau(z)| \sup_{t\in[0,t_0+1]} |a_x(t)|\right)$$

$$\leq \exp\left(\frac{(2-t_0)\epsilon}{2}\right)\exp\left(\frac{\epsilon}{2M}M\right)$$

$$= \exp(\log(\delta)) = \delta.$$

■

5.2 Sliced Tube

In this Subsection, besides introducing some useful new notation, we will state and prove Proposition 5.1.1, that will allow us to work with more convenient sets - the sliced tubes, instead of the standard tubular neighborhoods.

We say that $\mathcal{B} \subset \mathbb{R}^d$ is a *ball* if it is a convex compact set, symmetric about the origin.

Definition 5.2.1 (K-Ball) *Let $B(0,r)$ denote the Euclidean ball with radius r. Given $K > 1$ we say that a ball $\mathcal{B}$ is a K-ball if*

$$B(0,K^{-1}) \subset \mathcal{B} \subset B(0,K).$$

Remark 5.2.2 *Let $K > 1$ and let $\mathcal{B} \subset \mathbb{R}^d$ be a K-ball. Then there exists a norm $\|\cdot\|_{\mathcal{B}}$ such that*

1. $\mathcal{B} = \{v \in \mathbb{R}^d : \|v\|_{\mathcal{B}} \leq 1\}$;
2. $K^{-1}\|v\| \leq \|v\|_{\mathcal{B}} \leq K\|v\| \quad \forall v \in \mathbb{R}^d$.

Remark 5.2.3 *Notice that, since the cube $\mathcal{B} = [-1,1]^d$ is inscribed in a sphere with radius $\sqrt{d}/2$ and circumscribed on a sphere with radius $1/2$, the hypothesis of Remark 5.2.2 is satisfied for any $K > \max\{2, \sqrt{d}/2\}$ and the conclusion, in this case, holds with $\|\cdot\|_{\mathcal{B}}$ being the norm of the maximum.*

Remark 5.2.4 *Let $P^t = P_0^t$ be the family of linear isomorphisms induced by the linear Poincaré flow over the base-orbit of a (C, T, Δ)-model vector field. Notice that, by the sub-exponential growth of $\|P^t\|$ and Remark 5.2.3, we can conclude that, for all $t \in [0, T + \Delta]$, $P^t([-1, 1]^{d-2})$ is a K-ball of $\mathbb{R}^{d-2}$ with*

$$K = \max\left\{2, \frac{\sqrt{d-2}}{2}\right\} e^{C(T+\Delta)}.$$

Fix a model vector field $X \in \mathfrak{X}^1(U)$ and let $P_s^t : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ denote the linear isomorphism induced by the linear Poincaré flow on the segment $[s, t]$ of the orbit of zero (Definition 2.3.8). Define also $\beta_s^t \equiv |\langle P_s^t(e_d), e_d\rangle|$, where $e_d = (0, 0, \ldots, 0, 1)$. Given an interval $I = [a, b]$ on the real line, $\kappa \in (0, 1)$, $r > 0$ and a ball $\mathcal{B} \subset \mathbb{R}^{d-2}$ we define a sliced tube $\mathcal{S}_{\mathcal{B}}(\kappa, r, I)$ by

$$\mathcal{S}_{\mathcal{B}}(\kappa, r, I) \equiv \bigcup_{s \in I} (\{s\} \times P_a^s(r\mathcal{B} \times \{0\}) \times [-\kappa r\beta_a^s, \kappa r\beta_a^s]) .$$

We most frequently deal with a special type of sliced tube, for which the ball and the height of the slices depend on the left edge of I.

Definition 5.2.5 (Standard Sliced Tube) *The standard sliced tube is defined by*

$$\mathcal{S}(\kappa, r, I) \equiv \bigcup_{s \in I} \left(\{s\} \times P^s([-r, r]^{d-2} \times \{0\}) \times [-\kappa r\beta_s, \kappa r\beta_s]\right),$$

where $P^s = P_0^s$ and $\beta_s = |\langle P^s e_2, e_2\rangle|$.

Remark 5.2.6 *Notice that the standard sliced tube is a sliced tube with a change of scale in direction y. In fact, by the cocycle property of the linear Poincaré flow, we can deduce that*

$$\mathcal{S}(\kappa, r, [a, b]) = \mathcal{S}_{\mathcal{B}}(\beta_a\kappa, r, [a, b]),$$

where $\mathcal{B} = P^a([-1, 1]^{d-2} \times \{0\})$.

It is convenient to work with sliced tubes (instead of the usual tubular neighborhood) because these sets are saturated by a family of cross sections (the slices), each one being orthogonal to the base-orbit of the model vector field. The fact that the slices of the tube are intrinsically related to the linear Poincaré flow is also an important feature of these sets, since it makes possible the comparison between tubular neighborhoods and sliced tubes.

Proposition 5.2.7 (Approximation by sliced tubes) *Given $\Delta > 0$, $T > 2\Delta$, $C > 0$, $\lambda > 1$ and $K > 1$, there exist $\kappa_0 \in (0,1)$ and $\rho_0 > 0$ such that, for all (C,T,Δ)-model vector field X, for any $\kappa \in (0,\kappa_0)$, $\rho \in (0,\rho_0)$, for all interval $I = [a,b] \subset [0,T]$ satisfying $|I| > 2\Delta$ and for any K-ball $\mathcal{B} \subset \mathbb{R}^{d-2}$ we have*

$$\mathcal{S}_{\mathcal{B}}(\kappa, \rho\lambda^{-1}, [a, b-\Delta]) \subset \bigcup_{t\in[0,b-a]} \varphi^t(U_a) \subset \mathcal{S}_{\mathcal{B}}(\kappa, \rho\lambda, [a, b+\Delta]),$$

where $U_a = \{a\} \times \rho\mathcal{B} \times [-\kappa\rho, \kappa\rho]$.

Note that as $\Delta > 0$ approaches zero and as $\lambda > 1$ approaches 1, the better the approximation becomes. On the other hand, in order to achieve a good approximation of a tubular neighborhood by sliced tubes, we need to impose that the initial slice (U_a) is sufficiently small $(\rho \ll 1)$ and sufficiently thin $(\kappa \ll 1)$. Before proving Proposition 5.2.7, we state and prove a technical lemma which shows the relation between κ and the distortion of the sliced tube in the w-direction. This technical lemma is an adaptation of [AB1, Lemma 5] to our context.

Lemma 5.2.8 (Approximation of the linear part) *Given $\lambda > 1$, $T > 0$, $C > 0$ and $K > 1$ there exists $\kappa_0 \in (0,1)$ such that for any (C,T)-model vector field X, any $\kappa \in (0,\kappa_0)$ and for any K-ball $\mathcal{B} \subset \mathbb{R}^{d-2}$ we have*

$$(\lambda^{-1}P_s^t(\mathcal{B}\times\{0\}))\times[-\beta_t\kappa, \beta_t\kappa] \subset P_s^t(\mathcal{B}\times[-\kappa,\kappa]) \subset (\lambda P_s^t(\mathcal{B}\times\{0\}))\times[-\beta_t\kappa, \beta_t\kappa],$$

for all $[s,t] \in [0,T]$.

Proof: In order to simplify notation, we will write $P_s^t(\mathcal{B})$, instead of $P_s^t(\mathcal{B} \times \{0\})$.

By the invariance of the plane $\{y = 0\}$, we have, for all $s, t \in [0,T]$,

$$P_s^t(w,y) = \begin{pmatrix} A_s^t & B_s^t \\ 0 & \beta_s^t \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix}.$$

And in this case,

$$(P_s^t)^{-1} = P_t^s = \begin{pmatrix} (A_s^t)^{-1} & -(\beta_s^t)^{-1}(A_s^t)^{-1}B_s^t \\ 0 & (\beta_s^t)^{-1} \end{pmatrix}$$

Take

$$\kappa_0 = \frac{\lambda - 1}{K\lambda M_T},$$

where $M_T = e^{2CT}$. Also, observe that, by the sub-exponential growth of the linear flow over the base line of a model vector field,

$$\begin{aligned}\|(A_s^t)^{-1}B_s^t\| &\leq \|(A_s^t)^{-1}\| \cdot \|B_s^t\| \\ &\leq \|P_t^s\| \cdot \|P_s^t\| < e^{2C|t-s|} < e^{2CT}.\end{aligned}$$

Consider $0 < \kappa < \kappa_0$. First let us prove that

$$P_s^t(\mathcal{B} \times [-\kappa, \kappa]) \subset \lambda P_s^t(\mathcal{B}) \times [-\beta_s^t \kappa, \beta_s^t \kappa].$$

Assume that $w \in \mathcal{B}$ and $|y| \leq \kappa$. We need only to prove that

$$A_s^t w + B_s^t y \in \lambda P_s^t(\mathcal{B}).$$

Or, equivalently, that

$$\|(A_s^t)^{-1}(A_s^t w + B_s^t y)\|_{\mathcal{B}} < \lambda,$$

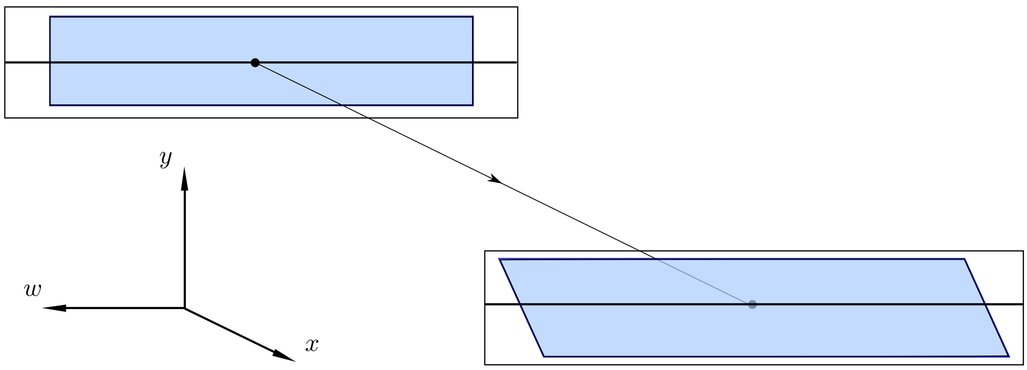

Figure 5.3: Distortion in the w-direction is not significant if the initial slice is thin enough.

Where $\|\cdot\|_{\mathcal{B}}$ is the norm given by Remark 5.2.2. Indeed,

$$\begin{aligned}\|(A_s^t)^{-1}(A_s^t w + B_s^t y)\|_{\mathcal{B}} &= \|w + (A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \\ &\leq \|w\|_{\mathcal{B}} + \|(A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \\ &\leq 1 + K\|(A_s^t)^{-1}B_s^t y\| \\ &\leq 1 + K\|(A_s^t)^{-1}B_s^t\| \cdot |y| \\ &\leq 1 + K \cdot M_T \kappa < \lambda,\end{aligned}$$

as we wanted to show. On the other hand, assume that

$$(\tilde{w}, \tilde{y}) \in \lambda^{-1}P_s^t(\mathcal{B}) \times [-\beta_s^t \kappa, \beta_s^t \kappa],$$

Let y be such that $|y| \leq \kappa$ and $w \in \mathcal{B}$ such that $\tilde{y} = \beta_s^t y$ and $\tilde{w} = \lambda^{-1}A_s^t w$. Then we have that

$$(P_s^t)^{-1}(\tilde{w}, \tilde{y}) = (\lambda^{-1}w - (A_s^t)^{-1}B_s^t y, y),$$

So we need only to prove that

$$\|\lambda^{-1}w - (A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \leq 1.$$

And indeed,

$$\begin{aligned}\|\lambda^{-1}w - (A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} &\leq \lambda^{-1}\|w\|_{\mathcal{B}} + \|(A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \\ &\leq \lambda^{-1} + K\|(A_s^t)^{-1}B_s^t y\| \\ &\leq \lambda^{-1} + K \cdot M_T \kappa \leq 1.\end{aligned}$$

■

Proof of Proposition 5.2.7:

Take any $1 < \bar{\lambda} < \lambda$ and choose $\kappa > 0$ as in Lemma 5.2.8, such that

$$\begin{aligned}\bar{\lambda}^{-1}P_a^s(\mathcal{B} \times \{0\}) \times [-\kappa\beta_a^s, \kappa\beta_a^s] &\subset P_a^s(\mathcal{B} \times [-\kappa, \kappa]) \\ &\subset \bar{\lambda}P_a^s(\mathcal{B} \times \{0\}) \times [-\kappa\beta_a^s, \kappa\beta_a^s],\end{aligned} \tag{5.8}$$

for all $[a, s] \in [0, T]$.

Let $\tau_s = \tau_{s-a,(a,0,0)}$ be the hitting-time function (Definition 2.1.2) with respect to the flow φ, with time $s-a$ and base point $(a,0,0)$.

Denote by N_s the subspace $N_{\varphi^s((a,0,0))} \subset T_{\varphi^s((a,0,0))}M$ orthogonal to $X(\varphi^s((a,0,0)))$ and let $\Phi^s : U_a \to N_s$ be the Poincaré Map with respect to the flow φ and time $s-a$, that is

$$\Phi^s(p) = \varphi^{\tau_s(p)}(p).$$

Choose $\rho > 0$ small enough so that if $p \in U_a = U_a(\kappa, \rho)$ then $|\tau_s(p) - s| < \Delta$ for all $s \in [a, T+\Delta]$, and in particular,

$$\tau_{\alpha-\Delta}(p) < \alpha < \tau_{\alpha+\Delta}(p) < \tau_{\beta-\Delta}(p) < \beta < \tau_{\beta+\Delta},$$

for all $a < \alpha < \beta < T$ with $\beta - \alpha > 2\Delta$. So

$$\bigcup_{s\in[\alpha+\Delta,\beta-\Delta]} \Phi^s(U_a) \subset \bigcup_{s\in[\alpha,\beta]} \varphi^s(U_a) \subset \bigcup_{s\in[\alpha-\Delta,\beta+\Delta]} \Phi^s(U_a). \tag{5.9}$$

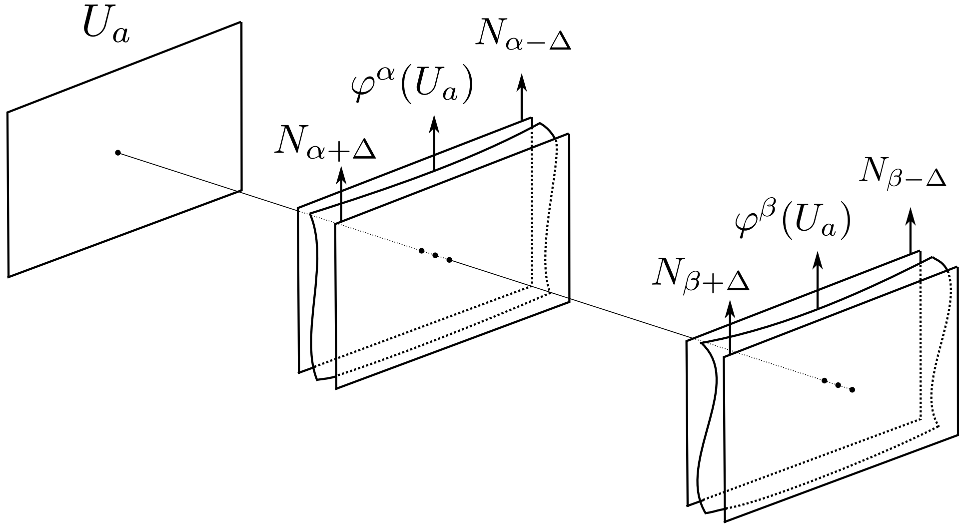

Figure 5.4: $|\tau_s(p) - s| < \Delta$ for all $s \in [a, T+\Delta]$.

Since

$$D\Phi^s_{(a,0,0)} = P^s_a \quad \text{(See Proposition 2.3.9)},$$

we can use the Taylor Formula to obtain

$$\Phi^s(p) = \underbrace{\Phi^s((a,0,0))}_{(a+s,0,0)} + P^s_a(p) + r(p)$$

where $r : U_0 \to \mathbb{R}^{d-1}$ satisfies

$$\lim_{\|p\|\to 0} \frac{\|r(p)\|}{\|p\|} = 0.$$

Notice that if r is identically zero then the conclusion of the proposition follows from (5.8) and (5.9). Taking a smaller ρ if necessary, the general case follows from Taylor approximation. ∎

Corollary 5.2.9 (Standard Approximation) *Given $\Delta > 0$, $T > 2\Delta$, $C > 0$ and $\lambda > 1$, there exist $\kappa_0 \in (0,1)$ and $\rho_0 > 0$ such that, for all (C, T, Δ)-model vector field X, for any $\kappa \in (0, \kappa_0)$, $\rho \in (0, \rho_0)$ and for all interval $I = [a, b] \subset [0, T]$ satisfying $|I| > 2\Delta$ we have*

$$\mathcal{S}(\kappa, \rho\lambda^{-1}, I_\Delta) \subset \bigcup_{t\in I} \varphi^t(U_0) \subset \mathcal{S}(\kappa, \rho\lambda, I^\Delta),$$

where $U_0 = \{0\} \times [-\rho, \rho]^{d-2} \times [-\kappa\rho, \kappa\rho]$.

Proof: Recall that, by Remark 5.2.6,

$$\mathcal{S}(\kappa, \rho, [a, b]) = \mathcal{S}_{\mathcal{B}_a}(\beta_a\kappa, \rho, [a, b]),$$

where $\mathcal{B}_a = P^a([-1, 1]^{d-2} \times \{0\})$. Let $\bar\kappa_0 \in (0, 1)$ and $\rho_0 > 0$ be given by Proposition 5.1.1 for the K obtained by Remark 5.2.4. The Corollary follows by taking

$$\kappa_0 = \bar\kappa_0 e^{-C(T+\Delta)}$$

and by observing that $\beta_a < \|P_0^a\| < e^{C(T+\Delta)}$. ∎

Remark 5.2.10 *Notice that if X is a C-model vector field with associated linear Poincaré flow given by*

$$P_s^t = \begin{pmatrix} A_s^t & B_s^t \\ 0 & \beta_s^t \end{pmatrix},$$

then the vector field given by

$$X_\epsilon(x, w, y) = X(x, w, y) - \epsilon y \cdot e_d$$

is a $(C + \epsilon)$-model vector field with the associated linear Poincaré flow given by

$$\widetilde{P}_s^t = \begin{pmatrix} A_s^t & \widetilde{B}_s^t \\ 0 & e^{-\epsilon(t-s)}\beta_s^t \end{pmatrix}.$$

Proposition 5.2.11 (Perturbed sliced tube) *Let X be model vector field, $\epsilon > 0$ and X_ϵ be as in the previous Remark, that is, $X_\epsilon(x, w, y) = X(x, w, y) - y\epsilon \cdot e_d$. Then, for any $\kappa \in (0, 1)$, $r > 0$ and any $I = [a, b] \subset [0, \infty)$, we have*

1. $\pi_x(\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)) = \pi_x(\mathcal{S}_{\mathcal{B}}(\kappa, r, I))$;
2. $\pi_w(\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)) = \pi_w(\mathcal{S}_{\mathcal{B}}(\kappa, r, I))$;
3. $\pi_y(\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)) \subset \pi_y(\mathcal{S}_{\mathcal{B}}(\kappa, r, I))$,

where $\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)$ denotes de sliced tube with respect to X_ϵ and π_x, π_w, π_y are the projections in directions x, w and y, respectively. In Particular,

$$\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I) \subset \mathcal{S}_{\mathcal{B}}(\kappa, r, I).$$

Proof: The first conclusion is trivial, since the projection in the coordinate x of a sliced tube is I. The second conclusion is a direct consequence of Remark 5.2.10, since

$$\tilde{P}_s^t|_{\{y=0\}} = A_s^t = P_s^t|_{\{y=0\}}.$$

In order to see that third consequence is also true, notice that, by Remark 5.2.10,

$$\tilde{P}_s^t|_{\{w=0\}} = e^{-\epsilon(t-s)}\beta_s^t < \beta_s^t,$$

since we are considering $0 < s < t$. ∎

5.3 Bump Function

In the previous subsection we saw that, in terms of volume, it is possible to approximate a tubular neighborhood by two sliced tubes: one that contains the neighborhood, and other that is contained in it. In this subsection, we show how to construct a bump function with small C^1 norm, supported on the inner (standard) sliced tube.

Remark 5.3.1 *Given a C-model vector field X, the function $\beta_t(x) = |\langle P_x^t(e_2), e_2\rangle|$, defined inside a sliced tube with respect to X is a linear cocycle in $\mathbb{R}$ over φ_X^t. Moreover, its logarithmic derivative is bounded by C. In other words, if we fix $x \in \mathbb{R}$ and let t vary, we have that*

$$\left|\frac{\beta_t(x)'}{\beta_t(x)}\right| \le C.$$

Proposition 5.3.2 (Bump Function) *Given $\epsilon > 0$ and $0 < \gamma < 1$ there exists $0 < \epsilon' < \epsilon$ such that for all $T_0, a, C > 0$ there exist $0 < \kappa_0 < 1$ and*

$r_0 > 0$ *such that for all* (C, T_0, a)*-model vector field* X *and for any* $T \in (2a, T_0)$, $\kappa \in (0, \kappa_0)$ *and* $r \in (0, r_0)$ *there exists a function* $\Psi : \mathbb{R}^d \to \mathbb{R}$ *such that*

1. $\Psi = 0$ *outside* $\mathcal{S}(\kappa, r, [-a, T + a])$*;*
2. $\Psi = \epsilon'$ *on* $\mathcal{S}(\kappa, r(1 - \gamma), [a, T - a])$*;*
3. $\|y\Psi(x, w, y)\|_{C^1} < \epsilon$*, for all* $(x, w, y) \in \mathbb{R}^d$*.*

Proof: Let $\epsilon' = \dfrac{\epsilon\gamma}{2 + \gamma}$ and fix any $C > 0$, any large $T_0 > 0$ and any small $a > 0$. Define

$$\kappa_0 = e^{-2C(T_0 + 2a)}$$

and take $T \in (2a, T_0)$, $\kappa \in (0, \kappa_0)$ and any (C, T_0, a)-model vector field X. Observe that, by the third property of model vector fields, by Remarks 2.6.1 and 5.3.1, we can conclude that

$$\begin{aligned}
(\beta_x \|(P^x)^{-1}\|)^{-1} &< \sup \beta_x^{-1} \cdot \sup \|(P^x)^{-1}\|^{-1} \\
&= (\inf \beta_x)^{-1} \cdot (\inf \|(P^x)^{-1}\|)^{-1} \\
&< e^{-2Cx} < e^{-2C(T_0+2a)} = \kappa_0.
\end{aligned}$$

Consider a bump-function $\xi_1 : \mathbb{R} \to \mathbb{R}$ such that:

- $\xi_1(x) = 0$, if $x \in \mathbb{R} \backslash [-1, 1]$;
- $\xi_1(x) = 1$, if $x \in [-1 + \gamma, 1 - \gamma]$;
- $|\xi_1'(x)| \le \frac{2}{\gamma}$, for all $x \in \mathbb{R}$.

Also define another bump-function $\xi_2 : \mathbb{R} \to \mathbb{R}$ such that

- $\xi_2(x) = 0$, if $x \in \mathbb{R} \backslash [-a, T + a]$;
- $\xi_2(x) = 1$, if $x \in [a, T - a]$,
- $|\xi_2'(x)| \le \frac{2}{a}$, for all $x \in \mathbb{R}$.

For some $r > 0$ (we will estimate r a posteriori) we define the bump-function

$$\begin{array}{rccc}
\Psi\colon & \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2} & \to & \mathbb{R} \\
& (x, w, y) & \mapsto & \epsilon' \xi_1(\kappa^{-1} r^{-1} \beta_x^{-1} y) \xi_1(r^{-1} \|(P^x)^{-1} w\|) \xi_2(x)
\end{array}$$

Properties 1 and 2 are very easy to check and we only need to estimate the derivative of $y\Psi(x, w, y)$ inside the tube $\mathcal{S}(\kappa, r, [-a, T + a])$. So in next computations we can always assume that

$$|y| < \kappa\beta_x r \tag{5.10}$$

and also that

$$\|(P^x)^{-1}w\| < r. \tag{5.11}$$

In order to prove that

$$\|y\Psi(x, w, y)\|_{C^1} < \epsilon,$$

we first observe that

$$|y\Psi(x, w, y)| < |y|\epsilon' < \epsilon.$$

Now let us compute the derivatives.

1.

$$\begin{aligned}
\left|\frac{\partial}{\partial y}(y\Psi(x, w, y))\right| &< |\Psi(x, w, y)| + |y| \left|\frac{\partial}{\partial y}(\Psi(x, w, y))\right| \\
&< \epsilon' + |y| \left(\frac{\epsilon'}{\kappa r\beta_x}|\xi_1'(\kappa^{-1}r^{-1}\beta_x^{-1}y)| \cdot |\xi_1(r^{-1}\|(P^x)^{-1}w\|)| \cdot |\xi_2(x)|\right) \\
&< \epsilon'\left(1 + \frac{2}{\gamma}\right) \\
&< \frac{\epsilon\gamma}{(\gamma + 2)}\frac{(\gamma + 2)}{\gamma} = \epsilon;
\end{aligned}$$

2.

$$\begin{aligned}
\|D_3(y\Psi(x, w, y))\| &< \epsilon'\frac{|y|}{r}|\xi_1'(r^{-1}\|(P^x)^{-1}w\|)| \cdot \|(P^x)^{-1}\| \\
&< \epsilon'\frac{\kappa r\beta_x}{r}\frac{2}{\gamma}\|(P^x)^{-1}\| \\
&< \epsilon'\frac{2}{\gamma}\kappa\beta_x\|(P^x)^{-1}\|,
\end{aligned}$$

By the choice of κ_0,

$$\kappa\beta_x\|(P^x)^{-1}\| < 1$$

and we have:

$$\begin{aligned}\|D_3(y\Psi(x,w,y))\| &< \epsilon'\frac{2}{\gamma}\\ &< \frac{\epsilon\gamma}{2+\gamma}\cdot\frac{2}{\gamma}\\ &< \frac{\epsilon}{1+\frac{\gamma}{2}} < \epsilon;\end{aligned}$$

3.

$$\begin{aligned}\left|\frac{\partial}{\partial x}(y\Psi(x,w,y))\right| &< \epsilon'|y|\cdot\left|\frac{\partial}{\partial x}\left(\xi_1(\kappa^{-1}r^{-1}\beta_x^{-1}y)\xi_1(r^{-1}\|(P^x)^{-1}w\|)\xi_2(x)\right)\right|\\ &< \epsilon'|y|\left|\left[\xi_1\left(\frac{y}{\kappa r\beta_x}\right)\right]'\right| + \left|\left[\xi_1\left(\frac{\|(P^x)^{-1}w\|}{r}\right)\right]'\right| + |\xi_2'(x)|\\ &< 2\epsilon'|y|\left(\frac{|y|}{\gamma\kappa r\beta_x^2}\cdot\frac{\partial\beta_x}{\partial x} + \frac{1}{\gamma r}\cdot\frac{\partial(\|(P^x)^{-1}w\|)}{\partial x} + \frac{1}{a}\right)\end{aligned}$$

By Remark 5.3.1, we have that $\left|\frac{\beta_x'}{\beta_x}\right| < C$ for all $x \in [-a, T_0+a]$. This estimate, together with the third property of model vector fields, gives us

$$\left|\frac{\partial}{\partial x}(y\Psi(x,w,y))\right| < 2\epsilon'\left(\frac{\kappa r\beta_x C}{\gamma} + \frac{\kappa r\beta_x C}{\gamma} + \frac{\kappa r\beta_x}{a}\right).$$

We can assume that $r > 0$ was chosen sufficiently small in order to obtain

$$\begin{aligned}\left|\frac{\partial}{\partial x}(y\Psi(x,w,y))\right| &< \frac{2\epsilon'}{\gamma}\\ &< \frac{\epsilon\gamma}{2+\gamma}\cdot\frac{2}{\gamma}\\ &< \frac{\epsilon}{1+\frac{\gamma}{2}} < \epsilon;\end{aligned}$$

Observe that the constant $r > 0$ did not influence the estimates of the derivative in the y and w directions and this scale invariance is expected when we work with C^1-perturbations. The fact that $r > 0$ is being used in the estimate of the derivative along the x-direction may seem strange, but notice that r is not related with the length (x-direction) of the sliced tube, but only with its transverse size. Indeed, the greater the length of the tube in relation to its thickness, less restrictive is the bump-function's derivative along the x-direction. ∎

5.4 Proof of the Fettuccine's Lemma

We will verify the conclusion of the Lemma for the pulled-back vector field F_*X and define a δ-crushable set U in $\mathbb{R}^d$ with respect to the C-sliced measure $\hat{m}$. In order to see that this Euclidean version of the Lemma is sufficient for the conclusion of the proof, notice that, since the perturbation is given by adding vertical vectors to the pulled-back vector field, Proposition 4.0.19 guarantees that the pushed-forward of the perturbation will be a $C\epsilon$-perturbation of the original vector field (recall that $C > 1$ depends only on the vector field $X \in \mathfrak{X}^3(M)$ and the cross section Σ). Moreover, since the C-sliced measure $\hat{m}$ is comparable to $F_*^{-1}(m)$, the relative δ-crushing property of $F_*(X)$ on U will originate a 4δ-crushing property in $F(U)$.

Let us fix $\epsilon > 0$ and $0 < \delta < 1$. Once defined the auxiliary constants

$$\gamma = 1 - \sqrt[d]{1 - \delta/2} \tag{5.12}$$

and

$$\delta' = (1 - \gamma)\delta, \tag{5.13}$$

we can choose $\epsilon' = \epsilon'(\epsilon, \gamma)$ from Proposition 5.3.2 (Bump Function) and $t_0 = t_0(\epsilon', \delta') > 1$, the crushing-time from Lemma 5.1.2. Now take any $T_0 > 3t_0$ and assume that $F : Z \to F(Z)$ is the tubular chart, given by Theorem 4.0.15 with respect to a non-periodic point $p \in \Sigma$ and time T_0. Notice that the sliced measure $\hat{m}$ with density ω of a standard sliced tube is given by

$$\hat{m}(\mathcal{S}(\kappa, r, I)) = 2\kappa r^{d-1} \int_I f(s) ds,$$

where

$$f(s) = |\det(P^s)| \cdot \omega(s)$$

and $P^s = P_0^s$ is the family of linear isomorphisms induced by the linear Poincaré flow. Observe that f has bounded logarithmic derivative (See §2.6), so we can use Proposition 2.6.2 to find a constant $a_0 = a_0(t_0, \gamma) \leq 1$ such that, for all interval $I \subset [0, T_0]$ with $|I| \geq t_0$ and for all $0 < a < a_0$,

$$\int_{I_a} f(s)ds > (1-\gamma) \int_{I^a} f(s)ds. \tag{5.14}$$

Recall that Property 10 of the tubular chart (Theorem 4.0.15) ensures that the bound of the logarithmic derivative of $f(s)$ depends only on the original vector field $X \in \mathfrak{X}^3(M)$. In fact, every time we evoke results of this Section about model vector fields, we will be implicitly using this Property.

Let $\kappa_0 = \kappa_0(T_0, a_0)$ and $\rho_0 = \rho_0(T_0, a_0)$ be as in Proposition 5.3.2 (Bump Function) and take $\lambda > 1$ such that

$$\lambda < \sqrt[3(d-1)]{\frac{1-\frac{\delta}{2}}{1-\delta}}. \tag{5.15}$$

With these choices of T_0, a_0 and λ, we can find $\kappa_1 > 0$ and $\rho_1 > 0$ from Corollary 5.2.9 (Standard approximation) in order to obtain, for any $I \subset [0, T_0]$, with $|I| \geq t_0$, for any $a \in (0, a_0)$, $\kappa \in (0, \kappa_1)$ and $\rho \in (0, \rho_1)$,

$$\mathcal{S}(\kappa, \rho\lambda^{-1}, I_a) \subset \bigcup_{s \in I} \varphi^s_X(U_0) \subset \mathcal{S}(\kappa, \rho\lambda, I^a), \tag{5.16}$$

where $U_0 = \{0\} \times [-\kappa\rho, \kappa\rho] \times [\rho, \rho]^{d-2}$. Let $K > 1$ be the constant given by Remark 5.2.4 and recall, that, by Remark 5.2.10, the vector field

$$X_{\epsilon'}(x, w, y) = X(x, w, y) - \epsilon' y \cdot e_d$$

is a $(C+\epsilon')$-model vector field. Thus, we can reduce κ_1 and ρ_1, if necessary, and use Proposition 5.2.7 to obtain, for any K-ball $\mathcal{B}$, any $I \subset [s, T]$ with $|I| \geq t_0$, any $a \in (0, a_0)$, $\kappa \in (0, \kappa_1)$ and $\rho \in (0, \rho_1)$,

$$\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, \rho\lambda^{-1}, [s, T-a]) \subset \bigcup_{t \in [0, T-s]} \varphi^t_{\widetilde{X}}(U_s) \subset \widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, \rho\lambda, [s, T+a]), \tag{5.17}$$

where $U_s = \{s\} \times (\rho\mathcal{B} \times \{0\}) \times [\kappa\rho, \kappa\rho]$.

Take

$$\kappa = e^{-C(T_0+a_0)} \min\{\kappa_0, \kappa_1\}.$$

Now, choose any $3t_0 < T < T_0$ and define $\rho_2 = \rho_2(T, a_0, \gamma)$ such that Lemma 5.1.2 (Crushing time) is satisfied. and $\rho = \min\{\rho_0, \rho_1, \rho_2\}$ and define a bump-function $\Psi = \Psi_{\lambda^{-1}\rho, \kappa}$ given by Proposition 5.3.2 such that

1. $\Psi \equiv 0$ outside $\mathcal{S}(\kappa, \lambda^{-1}\rho, [a, T-a])$;
2. $\Psi \equiv \epsilon'$ in $\mathcal{S}(\kappa, \lambda^{-1}(1-\gamma)\rho, [2a, T-2a])$;
3. $\|y\Psi\|_{C^1} < \epsilon$.

We define the perturbed vector field in $\mathbb{R}^d$ by

$$\widetilde{X}(x,w,y) \mapsto X(x,w,y) - y\Psi(x,w,y)$$

and denote the set to be crushed by

$$V = \mathcal{S}(\kappa, \rho\lambda^{-2}(1-\gamma), [2a, T-t_0-3a]).$$

Now let us verify that our choices were appropriate. First, note that, by relation (5.16) and by the construction of Ψ, $\widetilde{X}$ is indeed an ϵ-perturbation of X with support on

$$\mathcal{S}(\kappa, \rho\lambda^{-1}, [a, T-a]) \subset \bigcup_{t\in[0,T]} \varphi^t(U_0) = U.$$

Let us introduce some auxiliary notation:

$$\begin{aligned} S^- &= \mathcal{S}(\kappa, \rho\lambda, [-a, T-t_0+a]) \\ S^+ &= \mathcal{S}(\kappa, \rho\lambda^{-1}(1-\gamma), [t_0+a, T-2a]). \end{aligned}$$

Relation (5.16) ensures that $V \subset U^-$ and also that $U^- \subset S^-$. Therefore, we have that

$$\frac{\hat{m}(V)}{\hat{m}(U^-)} > \frac{\hat{m}(V)}{\hat{m}(S^-)} = \frac{2\kappa(\rho(1-\gamma)\lambda^{-2})^{d-1}\int_{2a}^{T-t_0-3a} f(s)ds}{2\kappa(\rho\lambda)^{d-1}\int_{-a}^{T-t_0+a} f(s)ds}.$$

The above Inequality, together with (5.12) , (5.14) and (5.15), leads us to

$$\begin{aligned} \frac{\hat{m}(V)}{\hat{m}(U^-)} &> \left(\frac{1-\gamma}{\lambda^3}\right)^{d-1} \cdot \frac{\int_{2a}^{T-t_0-3a} f(s)ds}{\int_{-a}^{T-t_0+a} f(s)ds} \\ &> \frac{(1-\gamma)^{d-1}}{\lambda^{3(d-1)}} \cdot \frac{\int_{2a}^{T-t_0-2a} f(s)ds}{\int_{-2a}^{T-t_0+2a} f(s)ds} \\ &> \frac{(1-\gamma)^d}{\lambda^{3(d-1)}} \\ &> \frac{1-\delta/2}{\frac{1-\delta/2}{1-\delta}} = 1-\delta. \end{aligned}$$

Claim 5.4.1

$$\bigcup_{t\in[0,t_0]} \varphi^t_{\widetilde{X}}(V) \in \mathcal{S}(\kappa, \rho\lambda^{-1}(1-\gamma), [a, T-2a]).$$

Proof of the Claim: Let $p \in V$. Then there exists $s \in [2a, T-t_0-3a]$ such that

$$p \in U_s = \{s\} \times P^s([-\rho', \rho']^{d-2} \times \{0\}) \times [-\kappa\rho'\beta_s, \kappa\rho'\beta_s],$$

where $\rho' = \rho(1-\gamma)\lambda^{-2}$. By the sub-exponential growth of β_s and the choice of κ,

$$\kappa\beta_s < \kappa e^{C(T+a)} < \kappa_1$$

and we can apply Relation (5.17) to deduce, together with Remark 5.2.10, that

$$\bigcup_{t\in[0,t_0]} \varphi^t_{\widetilde{X}}(U_s) \subset \widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa\beta_s, \lambda\rho', [s, s+t_0+a]) \subset \mathcal{S}_{\mathcal{B}}(\kappa\beta_s, \lambda\rho', [s, s+t_0+a]),$$

where $\mathcal{B} = P^s([-\rho', \rho'])$. By Remark 5.2.6,

$$\bigcup_{t\in[0,t_0]} \varphi^t_{\widetilde{X}}(p) \in \mathcal{S}(\kappa, \lambda\rho', [s, s+t_0+a]).$$

Since $2a < s < T-t_0-3_a$,

$$\varphi^t_{\widetilde{X}}(p) \in \mathcal{S}(\kappa, \rho(1-\gamma)\lambda^{-1}, [2a, T-2a])$$

and the Claim is proved. ■

The above claim we just proved together with relation (5.16) implies Item 3 of the Main Lemma. So we need only to prove item 4, that is,

$$\frac{\hat{m}(\varphi^{t_0}_{\widetilde{X}}(V))}{\hat{m}(U^+)} < \delta.$$

For that matter, let us define the set

$$W = \mathcal{S}(\delta'\kappa, (1-\gamma)\rho\lambda^{-1}, [t_0, T-2a]).$$

Since

$$\frac{\hat{m}(W)}{\hat{m}(S^+)} = \frac{\delta' \int_{t_0}^{T-2a} f(s)ds}{\int_{t_0+a}^{T-2a} f(s)ds}$$

$$< \frac{\delta' \int_{t_0-a}^{T} f(s)ds}{\int_{t_0+a}^{T-2a} f(s)ds}$$

$$< \frac{\delta'}{(1-\gamma)} = \delta$$

and since $S^+ \subset U^+$, for finishing the proof of the Lemma, it is sufficient to show that $\varphi_{\widetilde{X}}^{t_0}(V) \subset W$.

Note that Claim 5.4.1 placed us in the context of Lemma 5.1.2 (Crushing time) because

$$\psi = \epsilon' \quad \text{in} \quad \mathcal{S}(\kappa, \rho\lambda^{-1}(1-\gamma), [a, T-2a])$$

and, consequently, in this set, the perturbation $\widetilde{X}$ is the same as in Lemma 5.1.2.

Assume that $p_2 = (s, y_2, w_2) \in \varphi_{\widetilde{X}}^{t_0}(V)$ and let $p_1 = (r, y_1, w_1) \in V$ such that $\varphi_{\widetilde{X}}^{t_0}(p_1) = p_2$.

Let $\widetilde{\Phi} : N_{s-t_0} \to N_s$ be the Poincaré map with respect to the perturbed flow and $q \in N_{s-t_0}$ be such that $\widetilde{\Phi}(q) = p_2$, that is,

$$\varphi_{\widetilde{X}}^{\tilde{\tau}(q)}(q) = p_2.$$

Without loss of generality, we can assume that $\rho > 0$ is sufficiently small to guarantee that

$$|s - r - t_0| < a$$

and so that

$$t_0 < s < T - 2a. \tag{5.18}$$

By Claim 5.4.1, we have that

$$w_2 \in P^s([-\rho(1-\gamma)\lambda^{-1}, \rho(1-\gamma)\lambda^{-1}]^{d-2}) \tag{5.19}$$

and Lemma 5.1.2 implies that

$$\begin{aligned} |y_2| &= |\pi(\varphi_{\widetilde{X}}^{\tilde{\tau}(q)}(q))| \\ &< \delta' |\pi(\varphi^{\tau(q)}(q))| \\ &< \delta' \rho\kappa(1-\gamma)\lambda^{-1}\beta_s. \end{aligned}$$

The above Inequality together with (5.18) and (5.19) implies that $p_2 \in W$.

6
Global Crushing

Throughout the text we presented several ingredients necessary for the proof of Theorem 1.2.1. This section is devoted to combine all these pieces together. Before starting the proof, we will state a slightly different version of the "non-invariant Rokhlin lemma" from [AB1], which will be used in the scope of the proof.

Lemma 6.0.2 (Avila, Bochi) *Let $(\Lambda, \mathcal{K}, \sigma)$ be a Lebesgue space. Assume that $f : K \to K$ is a bi-measurable map which is continuous on an open subset with total measure and which is non-singular with respect to σ. Assume also that $\sigma(P_f) = 0$, where P_f is the set of periodic points for f. Then given any $\epsilon_0 > 0$ and $n, k \in \mathbb{N}$, with $k \leq n$, there exists an open set $B \subset \Lambda$ such that $f^{-i}(\bar{B}) \cap \bar{B} = \emptyset$ for $1 \leq i < n$,*

$$\sum_{i=0}^{n-1} \sigma(f^i(U)) > 1 - \epsilon_0 \quad \text{and} \quad \sum_{i=n-k-1}^{n-1} m(f^i(B)) < \frac{k}{n} + \epsilon_0.$$

Proof: Lemma 6.0.2 follows immediately from Theorem 2 and Remark 1 from [AB1] (Actually the result from [AB1] is more general because it deals with non-necessarily invertible maps). ∎

By Lemma 2.5.1 and Remark 2.5.4, we only need to show that, for any $0 < \delta < 1$, the set $\mathcal{V}_\delta$ is dense in the C^1-topology. So we fix $0 < \delta < 1$ and $X \in \mathfrak{X}^1(M)$ and explain how to construct a perturbation of X in $\mathcal{V}_\delta$. By Proposition 2.1.10, we can assume, without loss of generality, that X is a C^3 vector field with only hyperbolic *periodic orbits* (in particular the set of periodic orbits is finite).

For a fixed $\epsilon > 0$, we are going to show how to find $Y \in \mathcal{V}_\delta$ ϵ-close to X in the C^1 topology.

First, let $\Sigma \subset M$ be the transverse section given by Lemma 3.0.6 and (Λ, f, σ) its discrete induced dynamics. Take $t_0 = t_0(\epsilon, \delta) > 0$ as in Lemma 5.0.22. We will denote by $\delta_j = \delta_j(\delta) > 0$ the positive constant given by Remark

3.0.10, with $j \in \mathbb{N}$ in the role of n, δ_j in the role of $\delta > 0$ and with $\delta/5$ in the role of $\epsilon > 0$. That is,

$$\sigma(A) < \delta_j \Rightarrow m(\mathcal{T}_j(A)) < \frac{\delta}{5}. \tag{6.1}$$

Recall that there is a full-measure open subset G of the transverse section Σ where the first-return map f and the return-time function τ are continuous. Let G_n be the set of points in G that have n well-defined first-returns in G. Of course, $G_n \supset \Lambda$.

Let $\tau^- = \min_{x\in\Lambda}\{\tau(x)\}$, where $\tau : \Lambda \to \mathbb{R}_+$ is the return time function that defines f, and define

$$k = \left\lceil \frac{t_0}{\tau^-} \right\rceil \qquad \text{and} \qquad n = \left\lceil \frac{k}{\delta_1} \right\rceil.$$

Take $B \subset \Lambda$ the base of the tower given by Lemma 6.0.2 with respect to $k, n \in \mathbb{N}$ and $\delta_1/2$ (in the place of ϵ_0). So B is a relatively open set of Λ.

Define

$$T(x) = \sum_{j=0}^{n-1} \tau(f^j(x))$$

and notice that $T(x) > t_0$ for each $x \in B$. Also observe that, since $\sigma(G_n) < \infty$, there exists $T_0 > 0$ such that

$$A_{T_0} \equiv \{x \in G_n : T(x) < T_0\}$$

satisfies

$$\sigma(B \backslash A_{T_0}) < \frac{\delta_n}{2}. \tag{6.2}$$

Let $\kappa = \kappa(T_0)$ be given by Lemma 5.0.22. Consider the family $\mathcal{R}$ of all κ-rectangles R whose center is a non-periodic point $p \in B \cap A_{T_0}$, have diameter less than the number $\rho = \rho(p)$ provided by Lemma 5.0.22, and are contained in G_n (which is an open subset of the transverse section).

For each rectangle R in the family $\mathcal{R}$, we can apply Lemma 5.0.22 with $T = T(p) \in (t_0, T_0)$ (where p is the center of the rectangle) and obtain an ϵ-perturbation of X supported on the tube

$$U = U(R) = \bigcup_{t\in[0,T(p)]} \varphi^t(R).$$

For later use, let us compare the tube U with the tube

$$\mathcal{T}_n(R) = \bigcup_{x\in R} \bigcup_{t\in[0,T(x)]} \varphi^t(x).$$

Recall that R is contained in the set G_n where the function T is continuous. Therefore, if the diameter of R is small enough then $\mathcal{T}_n(R)$ and $U(R)$ are close in the following sense:

$$\frac{m(U(R)\triangle\mathcal{T}_n(R))}{m(U(R))} < \frac{\delta}{5}. \tag{6.3}$$

Reducing the family $\mathcal{R}$ if necessary, we assume that this property above holds for every $R \in \mathcal{R}$.

Now, since $\mathcal{R}$ is a Vitali cover of $B \cap A_{T_0}$ (Remark 4.0.21), there exists a finite number J of pairwise disjoint κ-rectangles R_j with diameter smaller than ρ_0, centered in non-periodic points p_j and such that

$$\sigma\left((B \cap A_{T_0})\setminus\bigcup_{j=1}^{J} R_j\right) < \frac{\delta_n}{2}. \tag{6.4}$$

Denote $U_j = U(R_j)$, the support of the j-th perturbation of X. By construction, the sets U_j have disjoint closures. Therefore, we can paste together the J perturbations and find a single ϵ-perturbation of X whose restriction to each U_j has the properties provide by Lemma 5.0.22, namely: There exists

$$V_j \subset U_j^- = \bigcup_{t\in[0,T-t_0]} \varphi^t(R_j)$$

such that

1. $\dfrac{m(V_j)}{m(U_j^-)} > 1-\delta;$

2. $\varphi_X^t(\overline{V_j}) \subset U_j \qquad \forall t \in [0,t_0];$

3. $\varphi_{\widetilde{X}}^t(\overline{V_j}) \subset U_j \qquad \forall t \in [0,t_0];$

4. $\dfrac{m(\varphi_{\widetilde{X}}^{t_0}(V_j)\triangle U_j^+)}{m(U_j^+)} < \delta.$

The compact set to be crushed is $K = \bigsqcup_j V_j$ and the perturbation is $\widetilde{X}$. Now we need to verify if K and $\widetilde{X}$ satisfy the crushing property, that is, if:

1. $m(M_R \cap K) > 1-\delta;$

2. $m(\varphi_{\widetilde{X}}^{t_0}(K)) < \delta.$

The second inequality is a direct consequence of Lemma 5.0.22. In fact, since $\varphi_{\widetilde{X}}^{t_0}(V_j) \subset U_j^+$, for all j and since the U_j's are pairwise disjoint, then

$$\varphi_{\widetilde{X}}^{t_0}\left(\bigsqcup_j V_j\right) = \bigsqcup_j \varphi_{\widetilde{X}}^{t_0}(V_j)$$

and, therefore,

$$m\left(\bigsqcup_j \varphi_{\widetilde{X}}^{t_0}(V_j)\right) \leq \frac{\delta}{5}\cdot m\left(\bigsqcup_j U_j^+\right) \leq \delta.$$

The first inequality will be verified in 4 steps:

1. $m\left(M_R\backslash\mathcal{T}_n(B)\right) < \delta/5$;
2. $m\left(\mathcal{T}_n(B)\backslash\bigsqcup_j U_j\right) < 2\delta/5$;
3. $m\left(\bigsqcup_j U_j\backslash\bigsqcup_j U_j^-\right) < \delta/5$;
4. $m\left(\bigsqcup_j U_j^-\backslash\bigsqcup_j V_j\right) < \delta/5$.

In order to verify (1), note that, by Lemma 6.0.2,

$$\sigma\left(\Lambda\backslash\bigcup_{i=0}^{n-1} f^i(B)\right) < \delta_1$$

and that, since

$$\mathcal{T}_1\left(\Lambda\backslash\bigcup_{i=0}^{n-1} f^i(B)\right) = M_R\backslash\mathcal{T}_n(B),$$

we have, by property (6.1), that $m(M_R\backslash\mathcal{T}_n(B)) < \frac{\delta}{5}$.

To verify the second inequality, first observe that $\bigsqcup_j R_j \subset B$ and, therefore,

$$\sigma\left(B\backslash\bigsqcup_j R_j\right) \leq \sigma(B\cap A_{T_0}^c) + \sigma\left((B\cap A_{T_0})\backslash\bigsqcup_j R_j\right).$$

From inequalities (6.2) and (6.4), we obtain

$$\sigma\left(B\backslash\bigsqcup_j R_j\right) \leq \delta_n.$$

So we can apply Equation (6.1) and conclude that

$$m\left(\mathcal{T}_n(B)\backslash\bigsqcup_j \mathcal{T}_n(R_j)\right) < \frac{\delta}{5}.$$

From Equation (6.3), we have that

$$m\left(\bigsqcup_j \mathcal{T}_n(R_j)\backslash\bigsqcup_j U_j\right) < \frac{\delta}{5}$$

and we conclude that

$$m\left(\mathcal{T}_n(B)\backslash\bigsqcup_j U_j\right) = m\left(\mathcal{T}_n(B)\backslash\bigsqcup_j \mathcal{T}_n(R_j)\right) + m\left(\bigsqcup_j \mathcal{T}_n(R_j)\backslash\bigsqcup_j U_j\right)$$
$$\leq \frac{\delta}{5} + \frac{\delta}{5} = \frac{2\delta}{5}.$$

The third one follows from Lemma 6.0.2. We only need to show that

$$U_j\backslash U_j^- \subset \mathcal{T}_1\left(\bigcup_{i=n-k-1}^{n-1} f^i(R_j)\right),$$

but this follows from the fact that we chose k such that $\varphi^{t_0}(p)$ hits B no more then k times, for all $p \in B$. Therefore, by Lemma 6.0.2,

$$\sigma\left(\bigcup_{i=n-k-1}^{n-1} f^i(R_j)\right) < \delta_1$$

and, then, by Equation (6.1),

$$m\left(\bigsqcup_j (U_j\backslash U_j^-)\right) < \frac{\delta}{5}.$$

Finally, the fourth inequality is a direct consequence of the Main Lemma (5.0.22) and the fact that the sets U_j are pairwise disjoint.

Without loss of generality, we could consider t_0 greater then the δ-crushing time for $M\backslash M_R$ (See Remark 2.5.2) and then, the δ-crushing property would be seen in the whole manifold M.

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