Metodologias Multicritérios de Apoio à Decisão

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A theoretical framework for Measuring Attractiveness by a Categorical Based Evaluation Technique (MACBETH)

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Introduction

Based on judgements of an evaluator D about the attractiveness of the elements of a finite set A of potential actions, MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique) is an approach to guide the construction of an interval scale for which the idea of difference of attractiveness is meaningful, that is for the construction of a numerical scale $v : A \to \Re : a \to v(a)$ which satisfies, not only the

Ordinal condition:

- $\forall\, a,b \in A,\ v(a) > v(b)$ if and only if D judges a more attractive than b,

but also the

Cardinal condition:

- $\forall a,b,c,d \in A$ with a more attractive than b and c more attractive than d, the ratio
$$\frac{v(a) - v(b)}{v(c) - v(d)}$$
expresses the relative strength for D of the difference of attractiveness between a and b taken as reference unit the difference of attractiveness between c and d.

To derive a scale satisfying the conditions above, the basic idea of MACBETH is:

1º) in a first stage, to use a very simple questioning procedure which involves only two actions in each question and to assign, to each element a of A, a real number $\mu(a)$ on the basis of straightforward rules for quantifying the preference information given by D;

2º) in a second stage, to discuss with D about the cardinality of the scale μ constructed in the first stage: does this scale satisfy the cardinal condition?

MACBETH has been first proposed in [Bana e Costa and Vansnick, 1994] and two applications in decision-aiding are described in [Bana e Costa and Vansnick, 1995]. This new paper presents a complete theoretical framework for our approach.

It includes three paragraphs: in the first one, we review the questioning procedure and indicate the measurement rules used in MACBETH; §2 and §3 are respectively devoted to the discussion of the existence and of the unicity of a solution for the representation problem we are addressing.

1. Questioning procedure and measurement rules

1.1. Questioning procedure

In the questioning procedure of MACBETH, an evaluator D is asked to make semantic judgements about the (subjective) difference of attractiveness between actions of A. The questioning mode appears like this:

$\forall(a,b) \in A \times A$ with $a \neq b$

	yes	no
Is a more attractive than b?	☐	☐

and, if the answer is yes:

Is the difference of attractiveness between a and b

very weak ?	☐
weak ?	☐
moderate?	☐
strong ?	☐
very strong ?	☐
extreme ?	☐

Concerning the responses of D, the following terminology and notations are adopted:

- We note aPb when D judges a more attractive than b (P is a binary relation on A);
- when aPb, we say that:

  (a,b) belongs to the category C_1 when D judges that the difference of attractiveness between a and b is **very weak** $[(a,b) \in C_1]$

  (a,b) belongs to the category C_2 when D judges that the difference of attractiveness between a and b is **weak** $[(a,b) \in C_2]$

  (a,b) belongs to the category C_3 when D judges that the difference of attractiveness between a and b is **moderate** $[(a,b) \in C_3]$

  (a,b) belongs to the category C_4 when D judges that the difference of attractiveness between a and b is **strong** $[(a,b) \in C_4]$

  (a,b) belongs to the category C_5 when D judges that the difference of attractiveness between a and b is **very strong** $[(a,b) \in C_5]$

(a,b) belongs to the category C_6 when D judges that the difference of attrac tiveness between a and b is **extreme** $[(a,b) \in C_6]$.

Remarks

① The categories C_1, C_2, C_3, C_4, C_5 and C_6 can be seen as six binary relations on A and we have:

$$P = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6.$$

② We will always suppose hereafter that the binary relations P, C_1, C_2, C_3, C_4, C_5 and C_6 are asymmetric and that, $\forall i, j \in \{1,2,3,4,5,6\}$ with $i \neq j$, $C_i \cap C_j = \varnothing$ Such conditions are supposed to be verified during the interaction with D.

③ The initial responses of D can be presented in a "matrix of judgements", corre sponding, in row and in column, to the elements of A, by filling in this matri according to the following rule: $\forall a,b \in A$ and $\forall k \in \{1,2,3,4,5,6\}$, the element a the intersection between the row corresponding to action a and the colum corresponding to action b is equal to:

$$\begin{array}{ll} 0 & \text{if and only if (not } aPb) \\ k & \text{if and only if } (a,b) \in C_k \end{array}$$

1.2. Measurement rules

In order to numerically represent the qualitative information given by D, som measurement rules are of course necessary. Two basic rules are used in MACBETH for assigning, to each element a of A, a real number $\mu(a)$:

Rule 1

$$\forall a,b \in A: \ aPb \Leftrightarrow \mu(a) > \mu(b)$$

that is, the number assigned to action a is strictly greater than the number assigned to action b if and only if, for D, a is more attractive than b.

Rule 2

$$\forall k,k' \in \{1,2,3,4,5,6\} \text{ with } k \neq k' \text{ and } \forall a,b,c,d \in A \text{ with } (a,b) \in C_k \text{ and } (c,d) \in C_{k'}:$$
$$k > k' \ \Leftrightarrow \ \mu(a) - \mu(b) > \mu(c) - \mu(d)$$

that is, when aPb and cPd, with (a,b) and (c,d) not belonging to the same cate gory, the difference between the numbers associated to a and b is strictly greate than the difference between the numbers associated to c and d if and only if the dif ference of attractiveness between a and b is judged greater than the difference o attractiveness between c and d (note that this information is not asked directly to D but it follows indirectly from the matrix of judgements).

The justification for including the constraint $k \neq k'$ in rule 2 is that, when (a,b) and (c,d) are assigned the same category, there is no reason to assume that $\mu(a)-\mu(b)=\mu(c)-\mu(d)$.

2. Existence of a solution

In this paragraph, we address the following problems: 1) what are the necessary and sufficient conditions for the existence of a scale $\mu : A \rightarrow \Re$ satisfying rules 1 and 2, and 2) how to verify in the practice if these conditions are satisfied?

A main part of the answer to the first problem derives from:

Proposition 1

$\exists\, \mu : A \rightarrow \Re$ satisfying rules 1 and 2

if and only if

$\exists\, \mu : A \rightarrow \Re$ and six real numbers t_1, t_2, t_3, t_4, t_5 and t_6

satisfying the following conditions:

$$0 = t_1 < t_2 < t_3 < t_4 < t_5 < t_6$$

and

$$\forall a,b \in A \text{ and } \forall k \in \{1,2,3,4,5,6\},\ \mu(a) > \mu(b) + t_k \Leftrightarrow a P_k\, b,$$

where $\forall a,b \in A$ and $\forall k \in \{1,2,3,4,5,6\}$: $a P_k\, b \Leftrightarrow (a,b) \in C_k \cup C_{k+1} \cup \ldots \cup C_6$

The proof of proposition 1 is almost obvious given the property of "continuity" of the set of real numbers and the fact that A is a finite set. Proposition 1 is important because it shows that, from a theoretical point of view, the problem we have to solve is similar to the problem of the constant threshold representation of a m-tuple of binary relations. This problem was solved by Doignon [1987] and his main result is:

Let $R = (R_1, R_2, \ldots, R_m)$ be an m-tuple of binary relations on a given finite set X. A cycle from R is any sequence of pairs of the form (x_1,x_2), (x_2,x_3), ..., (x_{j-1},x_j), (x_j,x_1), all of them taken in $R_1 \cup R_1^{dual} \cup R_2 \cup R_2^{dual} \cup \ldots \cup R_m \cup R_m^{dual}$. A p-cyclone from R is any non-empty union of at most p cycles from R. When, for each $j = 1,2,\ldots,m$, the same number (possibly zero) of pairs is taken in R_j and R_j^{dual}, the cyclone is called "balanced".

Theorem 1 (proposition 4 in [Doignon, 1987, page 80])

Assume $m \geq 2$. There are a real-valued mapping f on X and real numbers $\sigma_1, \sigma_2, \ldots, \sigma_m$ such that, for all $j \in \{1,2,\ldots,m\}$ and $x,y \in X$: $xR_j y \Leftrightarrow f(x) > f(y) + \sigma_j$

if and only if

no m-cyclone from R is balanced.

With respect to this general case, our problem presents the following particularities:

1) the thresholds t_1, t_2, t_3, t_4, t_5 and t_6 have to be non negative

and

$\forall k, k' \in \{1,2,3,4,5,6\}$ with $k < k'$: $t_k < t_{k'}$

that is the relations P_k have to be irreflexive and $P_1 \supset P_2 \supset P_3 \supset P_4 \supset P_5 \supset P_6$ (Note that these conditions are always satisfied taking into account the questioning procedure and the definition of the relations P_k.)

2) $t_1 = 0$ that is

(2.1) $P_1 (\equiv P)$ has to be a strict weak order (an asymmetric and negatively transitive relation)

and

(2.2) $\forall a,b,c,d \in A$ with
(not aPb), (not bPa), aPc and dPa and $\forall k,k' \in \{1,2,3,4,5,6\}$,
we must have
$[(a,c) \in C_k \Leftrightarrow (b,c) \in C_k]$ and $[(d,a) \in C_{k'} \Leftrightarrow (d,b) \in C_{k'}]$.

When the relations P_k are irreflexive and $P_1 \supset P_2 \supset P_3 \supset P_4 \supset P_5 \supset P_6$ (which results from remark ② above and will be always be assumed hereafter), it can be proven that the cyclone condition of Doignon, (2.1) and (2.2) are necessary and sufficient conditions for the existence of a scale $\mu : A \to \Re$ satisfying rules 1 and 2. It is interesting to observe that, if we adopt the following definitions

"a generalised cycle from $\mathcal{P} = (P_1, P_2, P_3, P_4, P_5, P_6)$ is any sequence of pairs of the form (a,x_2), (x_2,x_3), ..., (x_{j-1},x_j), (x_j,b), all taken in $P_1 \cup P_1^{dual} \cup P_2 \cup P_2^{dual} \cup \ldots \cup P_6 \cup P_6^{dual}$ with (not aP_1b) and (not bP_1a);
a generalised p-cyclone from $\mathcal{P}$ is any non-empty union of a most p generalised cycles from $\mathcal{P}$;
when, for each $j = 1,2,\ldots,m$, the same number (possibly zero) of pairs is taken in P_j and P_j^{dual}, the generalised cyclone is called balanced",

the following result can be stated:

<u>Theorem 2</u>

$\exists\, \mu : A \to \Re$ satisfying rules 1 and 2

if and only if

no generalised n-cyclone from $\mathcal{P} = (P_1, P_2, P_3, P_4, P_5, P_6)$ is balanced

where n is the number of non-empty relations among C_1, C_2, C_3, C_4, C_5 and C_6.

From a practical point of view, it is of course difficult to test the "cyclone condition". However, it is possible to verify the existence of a solution for our problem thanks to the following linear program (LP1), the variables of which are $\mu(a)$ $(a \in A), s_1, s_2, s_3, s_4, s_5, s_6$ and ℓ:

Min ℓ

s.t. each variable ≥ 0

$s_1 = 0$

$\forall k \in \{2,3,4,5,6\}$: $s_k \geq s_{k-1} + \Delta$ (where Δ is any real constant ≥ 2)

$\forall k \in \{1,2,3,4,5,6\}$ and $\forall (a,b) \in C_k : \mu(a) - \mu(b) \geq s_k + 1 - \ell$

$\forall k \in \{1,2,3,4,5\}$ and $\forall (a,b) \in C_k : \mu(a) - \mu(b) \leq s_{k+1} - 1 + \ell$

$\forall (a,b) \notin P : \mu(a) \leq \mu(b)$.

Indeed, taking into account that, if μ satisfies the rules 1 and 2, then, for each $\theta_1 \in \Re_0^+$ and for each $\theta_2 \in \Re$, $\mu^* = \theta_1.\mu + \theta_2$ also satisfies these rules, it is easy to prove that, for any real constant $\Delta \geq 2$:

Proposition 2

$\exists\, \mu : A \to \Re$ satisfying rules 1 and 2

if and only if

$\exists\, \mu : A \to [0,+\infty[$ and six real numbers s_1, s_2, s_3, s_4, s_5 and s_6

satisfying the following conditions

Condition 1: $s_1 = 0$

Condition 2: $\forall k \in \{2,3,4,5,6\}$: $s_k \geq s_{k-1} + \Delta$

Condition 3: $\forall k \in \{1,2,3,4,5,6\}$ and $\forall (a,b) \in C_k : \mu(a) - \mu(b) \geq s_k + 1$

Condition 4: $\forall k \in \{1,2,3,4,5\}$ and $\forall (a,b) \in C_k : \mu(a) - \mu(b) \leq s_{k+1} - 1$

Condition 5: $\forall (a,b) \notin P$: $\mu(a) \leq \mu(b)$

if and only if

the optimal solution ℓ_{min} of LP1 is equal to 0.

From proposition 2 and theorem 2, it follows that $[\ell_{min} > 0]$ implies that there exists (at least) one generalised cyclone from $\mathcal{P} = (P_1, P_2, P_3, P_4, P_5, P_6)$ which is balanced. (Note that LP1 is always feasible.)

It is interesting to observe that such a cyclone can be found from the optimal solution of the following linear program (LP2), the variables of which are $\mu(a)$ $(a \in A)$, $s_1, s_2, s_3, s_4, s_5, s_6$, $\alpha(a,b)$ and $\delta(a,b)$ $[(a,b) \in C_1 \cup C_2 \cup \ldots \cup C_6]$, $\beta(a,b)$ and $\gamma(a,b)$ $[(a,b) \in C_1 \cup C_2 \cup \ldots \cup C_5]$:

$$\text{Min}\left[\sum_{(a,b)\in C_1\cup C_2\cup\ldots\cup C_6}\alpha(a,b)+\sum_{(a,b)\in C_1\cup C_2\cup\ldots\cup C_5}\beta(a,b)\right]$$

s.t. each variable ≥ 0

$s_1 = 0$

$\forall k \in \{2,3,4,5,6\}$: $s_k \geq s_{k-1} + \Delta$

$\forall k \in \{1,2,3,4,5,6\}$ and $\forall(a,b) \in C_k$: $\mu(a) - \mu(b) \geq s_k + 1 - \ell_{min}$

$\forall k \in \{1,2,3,4,5\}$ and $\forall(a,b) \in C_k$: $\mu(a) - \mu(b) \leq s_{k+1} - 1 + \ell_{min}$

$\forall(a,b) \notin P$: $\mu(a) \leq \mu(b)$

$\forall k \in \{1,2,3,4,5,6\}$ and $\forall(a,b) \in C_k$: $\mu(a) - \mu(b) = s_k + 1 - \alpha(a,b) + \delta(a,b)$

$\forall k \in \{1,2,3,4,5\}$ and $\forall(a,b) \in C_k$: $\mu(a) - \mu(b) = s_{k+1} - 1 + \beta(a,b) - \gamma(a,b)$.

Let us point out that, when $\alpha(a,b)$ (or $\beta(a,b)$) is not equal to zero in the optimal solution of LP2, it indicates that this problem did not succeed in trying to satisfy the condition 3 or 4 corresponding to (a,b) in proposition 2, that is:

$$s_k + 1 \leq \mu(a) - \mu(b) \leq s_{k+1} - 1 \text{ when } (a,b) \in C_k \quad (k = 1,2,3,4,5)$$
$$\text{or } s_6 + 1 \leq \mu(a) - \mu(b) \text{ when } (a,b) \in C_6$$

(see figure 1).

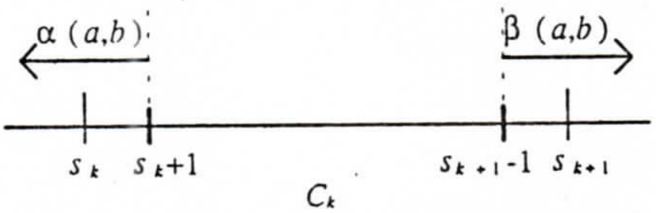

Figure 1: Variables $\alpha(a,b)$ and $\beta(a,b)$ for $k < 6$

Example: Let us consider the following matrix of judgements

	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1
a_8	0	4	5	5	5	6	6	6
a_7	0	0	1	2	3	4	5	6
a_6	0	0	0	2	2	4	4	5
a_5	0	0	0	0	2	2	4	4
a_4	0	0	0	0	0	2	3	4
a_3	0	0	0	0	0	0	2	2
a_2	0	0	0	0	0	0	0	1
a_1	0	0	0	0	0	0	0	0

In terms of the variables $\alpha(a,b)$ and $\beta(a,b)$, the optimal solution of LP2 is:

α	a_1	a_4	a_5	a_6
a_5		1		
a_6			1	
a_7	1			
a_8				1

β	a_1	a_2	a_4	a_6
a_2	1			
a_5	1			
a_6		1		
a_7				1
a_8			1	

As shown by figure 2, these values allow to determine both the following cycles from $\mathcal{P} = (P_1, P_2, P_3, P_4, P_5, P_6)$:

$$(a_8,a_6)\ (a_6,a_5)\ (a_5,a_4)\ (a_4,a_8)$$

and

$$(a_7,a_1)\ (a_1,a_2)\ (a_2,a_6)\ (a_6,a_7)$$

the union of which is a balanced cyclone.

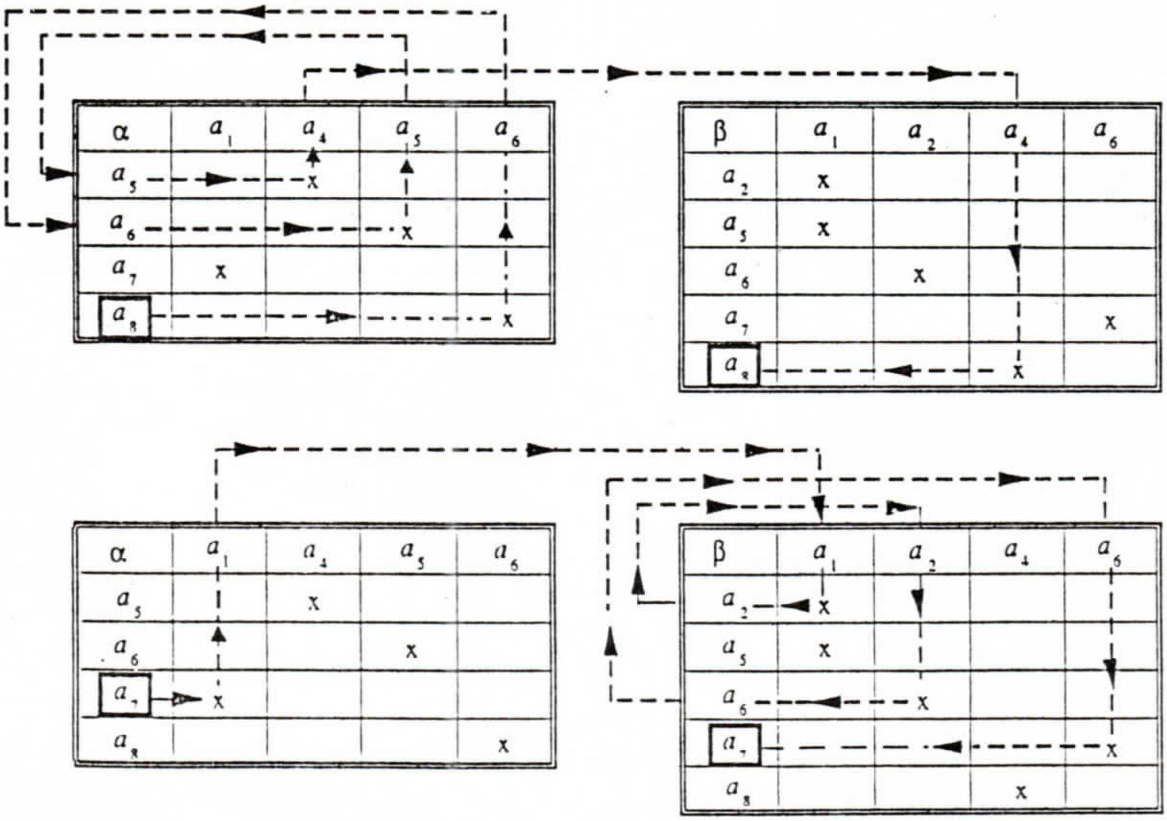

Figure 2: Determination of balanced cyclones from the values $\alpha(a,b)$ and $\beta(a,b)$ which are not equal to zero in the optimal solution of LP2

3. Unicity of the solution

When there exists a scale $\mu : A \rightarrow \Re$ satisfying the rules 1 and 2, such a scale is no unique: indeed, these rules are satisfied not only by $\mu^* = \theta_1.\mu + \theta_2$ for each $\theta_1, \theta_2 \in \Im$ with $\theta_1 > 0$, but also by others scales.

Example

Let us consider the following matrix of judgements

	a_5	a_4	a_3	a_2	a_1
a_5	0	1	3	5	6
a_4	0	0	2	4	5
a_3	0	0	0	2	3
a_2	0	0	0	0	1
a_1	0	0	0	0	0

The following scales μ' and μ'' both satisfy rules 1 and 2; nevertheless there are no θ_1 and θ_2 such that $\mu'' = \theta_1.\mu' + \theta_2$.

$\mu'(a_5) = 100$	$\mu''(a_5) = 100$
$\mu'(a_4) = 75$	$\mu''(a_4) = 95$
$\mu'(a_3) = 45$	$\mu''(a_3) = 55$
$\mu'(a_2) = 1$	$\mu''(a_2) = 20$
$\mu'(a_1) = 0$	$\mu''(a_1) = 0$

How to determine one scale between all possible ones?

Taking into account the aim of MACBETH, we propose to determine a scale μ on the basis of the following general principle:

Let
- a^- be an element of A such that, $\forall a \in A$, not $a^- P\, a$

and
- a^+ be an element of A such that, $\forall a \in A$, not $a\; P\; a^+$

(these elements necessarily exist when rule 1 is satisfied, and they are different when P is a non-empty relation, which will be assumed hereafter).

Among all the possible scales μ with $\mu(a^-) = 0$ and $\mu(a^+) = 100$, our basic idea is to favour a numerical representation such that:

1) the differences of attractiveness corresponding to pairs of actions assigned to the same category are as close as possible, and
2) the differences of attractiveness corresponding to pairs assigned to different categories are as far as possible.

Technically, our first idea to meet this general principle was to minimise, under appropriate constraints, the sum of the distances between each $[\mu(a)-\mu(b)]$ and the centre $(s_k+s_{k+1})/2$ of the interval corresponding to the category C_k $(k=1,2,3,4,5)$ to which the pair (a,b) belongs (see [Bana e Costa and Vansnick, 1994]).

Another way to make operational the basic idea above stated consists in solving the following linear program (LP3), the variables of which are $\mu(a)$ $(a \in A)$, $s_1, s_2, s_3, s_4, s_5, s_6$ and ℓ:

Max ℓ

s.t. each variable ≥ 0

$$s_1 = 0$$
$$\mu(a^-) = 0$$
$$\mu(a^+) = 100$$
$$\forall k \in \{2,3,4,5,6\} : s_k \geq s_{k-1} + 2.\ell$$
$$\forall k \in \{1,2,3,4,5,6\} \text{ and } \forall (a,b) \in C_k : \mu(a) - \mu(b) \geq s_k + \ell$$
$$\forall k \in \{1,2,3,4,5\} \text{ and } \forall (a,b) \in C_k : \mu(a) - \mu(b) \leq s_{k+1} - \ell$$
$$\forall (a,b) \notin P : \mu(a) \leq \mu(b).$$

When applied to the matrix of judgements of the example above, LP3 gives, in terms of μ, the following results:

$$\mu(a_5) = 100$$
$$\mu(a_4) = 83,333$$
$$\mu(a_3) = 50$$
$$\mu(a_2) = 16,667$$
$$\mu(a_1) = 0.$$

With this scale, all the differences of attractiveness corresponding to pairs of actions assigned to the same category are equal, and the differences of attractiveness corresponding to pairs of actions assigned to different categories are at least 16,667 away. Let us point out that there are other scales for which all the differences of attractiveness corresponding to pairs of actions assigned to the same category are equal, but in these other scales the minimal "distance" between differences of attractiveness corresponding to pairs of actions assigned to different categories is then less than 16,667.

Example:

$$\mu^*(a_5) = 100$$
$$\mu^*(a_4) = 85$$
$$\mu^*(a_3) = 50$$
$$\mu^*(a_2) = 15$$
$$\mu^*(a_1) = 0.$$

Remark

When P is a (non-empty) strict weak order, there exists a scale satisfying rules 1 and 2 if and only if the optimal solution ℓ_{max} of LP3 is > 0. Consequently, LP3 could also be used to test the existence of a solution for the problem we are studying, when it is known that P is a strict weak order.

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