ZERMELO'S AXIOM OF CHOICE

Its Origins, Development, & Influence

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When Zermelo defended the Axiom of Choice by axiomatizing set theory in 1908, a number of his critics attacked that axiomatization. The next decade saw the Axiom explored by various mathematicians in algebra and analysis, but rarely exploited to the full. The situation was profoundly altered when Wacław Sierpiński founded the Warsaw school of mathematics soon after Poland's reunification in 1918. For in Sierpiński, the Axiom gained an advocate who investigated its relationship to many branches of mathematics and who successfully encouraged his students to do likewise. Meanwhile in Germany, Abraham Fraenkel began to study models of Zermelo's axiomatization and to establish the independence of the Axiom, provided that infinitely many urelements were allowed.[7] In 1938 Kurt Gödel obtained deep results about models of set theory by proving the relative consistency of both the Axiom of Choice and the Generalized Continuum Hypothesis. In particular he showed that every model of the usual postulates for set theory, but not necessarily of the Axiom, has a submodel in which both the Axiom of Choice and the Generalized Continuum Hypothesis are true. For all but the most radical constructivists, Gödel's result dispelled any suspicions that the Axiom might lead to a contradiction.

During the quarter-century after Gödel's work, the Axiom was fruitfully applied in diverse fields of mathematics, and many kinds of propositions were shown to be equivalent to it.[8] Moreover, mathematicians studied various propositions which depended on the Axiom but which were suspected of being weaker than it, such as the Prime Ideal Theorem for Boolean algebras. Nevertheless, there was no known way to determine completely the relative strength of these propositions when they involved real numbers, such as the existence of a non-measurable set. This defect in the Fraenkel–Mostowski method of independence proofs was remedied by Paul Cohen's method of forcing, which in 1963 enabled him to demonstrate the independence of the Axiom for systems of set theory lacking urelements. At the same time he established the independence of the Continuum Hypothesis in first-order logic. Set theorists began at once to exploit Cohen's technique and to reformulate it in a general setting. The next two decades produced a cornucopia of independence results involving the Axiom as well as other assumptions. Indeed, the primary focus of research no longer lay within set theory proper, but increasingly centered on *models* of set theory.

However, one consequence of Cohen's method was not altogether welcome. As such semantic investigations of set theory proliferated, the Axiom of Choice and other axioms of set theory (beyond those of Zermelo–Fraenkel) came to be regarded more like the axioms for a group, which has many models, than like the postulates for a categorical system such as the natural numbers.

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[7] An *urelement* (also called an individual or an atom) was an object which contained no elements, belonged to some set, and yet was not identical with the empty set.

[8] See Appendix 2 as well as Rubin and Rubin 1963.

Since set theory had provided a foundation for mathematics, the unity of mathematics was threatened. In 1967 Andrzej Mostowski commented on this matter:

> Such [post-Cohen independence] results show that axiomatic set theory is hopelessly incomplete .... Of course if there are a multitude of set-theories, then none of them can claim the central place in mathematics. Only their common part could claim such a position; but it is debatable whether this common part will contain all the axioms needed for a reduction of mathematics to set theory. [1967, 93–95]

A decade later Jean Dieudonné added:

> The proof by Gödel and P. Cohen that the Axiom of Choice and the Continuum Hypothesis are undecidable, and the numerous metamathematical works which resulted, have greatly changed the views of many mathematicians .... Beyond classical analysis (based on the Zermelo–Fraenkel axioms supplemented by the *Denumerable* Axiom of Choice), there is an *infinity* of different possible mathematics, and for the time being no definitive reason compels us to choose one of them rather than another. [1976, 11]

It remains to be seen whether the Axiom of Constructibility, Martin's Axiom, or some large cardinal axiom will eventually restore the unity of set theory and, with it, the foundations of mathematics.

Chapter 1

The Prehistory of the Axiom of Choice

> But as one cannot apply infinitely many times an *arbitrary* rule by which one assigns to a class A an individual of this class, a *determinate* rule is stated here.
>
> Giuseppe Peano [1890, 210]

Throughout its historical development, mathematics has oscillated between studying its assumptions and studying the objects about which those assumptions were made. After the introduction of new mathematical objects, it often happened that the assumptions underlying them remained unspecified for a considerable time; only through extensive use did such assumptions become sufficiently clear to receive an explicit formulation. Usually a body of theorems, consequences of an assumption, were obtained before the assumption itself came to be recognized. At times, indeed, an assumption was specified precisely in order to secure a particular theorem or theorems. Of course, such an assumption ordinarily formed part of a nexus of suppositions with varying degrees of explicitness. What, one may ask, has caused such an assumption to become conscious and explicit? The question grows more complex as soon as we recognize that there was rarely, if ever, a single way of expressing an assumption and that various weakenings or strengthenings of an assumption could serve different mathematical purposes. This preliminary chapter explores how the use of arbitrary choices led, over most of a century, to Zermelo's explicit formulation of the Axiom of Choice.

1.1 Introduction

In 1908 Zermelo proposed a version of the Axiom of Choice that is useful for describing weaker assumptions:

(1.1.1) Given any family T of non-empty sets, there is a function f which assigns to each member A of T an element $f(A)$ of A. [1908a, 274]

Such an f is now called a choice function for T. If (1.1.1) is limited to those families T of a particular cardinality, one obtains a restricted form of the Axiom. Since for any finite T the Axiom is provable, the weakest non-trivial case occurs when T is denumerable.[1] This case is known as the Denumerable Axiom of Choice, and is abbreviated hereafter as the Denumerable Axiom.[2]

At that time Zermelo regarded the Axiom of Choice as codifying an assumption that numerous mathematicians had already made implicitly [1908, 113]. One may then inquire how this assumption developed from earlier mathematics and through what stages it passed on its way to Zermelo's explicit formulation. To answer this question, we need a precise criterion for deciding what constituted an implicit use of the Axiom. The nature of such a use was different during the period before Zermelo's formulation than it was afterward. For this reason the author has termed such an implicit use prior to September 1904 as an "implicit use of the Assumption." Likewise, such an implicit use of the Denumerable Axiom will be called an implicit use of the Denumerable Assumption. After 1904, mathematicians were often conscious that making infinitely many arbitrary choices brought the Axiom into play. Before that date, a mathematician who made such choices seldom recognized that he had done anything unusual or questionable (see 1.2–1.7). Apparently the only exceptions were three Italian mathematicians who began to avoid such choices intentionally during the period 1890–1902 (see 1.8). The earliest of these was Peano, whose views were quoted at the beginning of this chapter.

The hallmark of an implicit use of the Assumption was the occurrence of infinitely many arbitrary choices. Sometimes these choices occurred quite explicitly, sometimes less so. Occasionally a mathematician would employ a proposition which had been proved, by himself or another, using an infinity of arbitrary choices and for which no other proof was then known. Although that mathematician may not have recognized that he had made such arbitrary selections indirectly, we have regarded this case too as an implicit use of the Assumption.

Some uses of the Axiom will be termed avoidable, others unavoidable. Briefly, an avoidable use in a proof was such that the given proof could be modified, with the techniques then available, to specify uniquely whatever the Axiom had been used to select. Sometimes, while the proof could not be modified in this manner, another demonstration of the same theorem was later found which did not rely on such choices. Whenever such is the case, it is noted in the text. Of course, the fact that an implicit use of the Assumption is avoidable does not imply in any way that the mathematician involved should have revised his proof.

An unavoidable use of the Axiom was one such that the proposition in

[1] A set is *denumerable* if it can be mapped one-one onto the set $\mathbb{N}$ of natural numbers.

[2] The Denumerable Axiom of Choice has also been called the Countable Axiom of Choice (see Jech 1973, 20). However, we distinguish sharply between a denumerable set and one that is *countable*, *i.e.*, finite or denumerable.

question could not be proved in Zermelo–Fraenkel set theory without urelements (hereafter called **ZF**), or in Zermelo–Fraenkel set theory with urelements (hereafter termed **ZFU**), but could be deduced in **ZF** supplemented by the Axiom of Choice (now known as **ZFC**). At times we refer to an unavoidable use by saying that a proposition P needs or requires the Axiom, though strictly speaking this is an *abus de langage*. When it is now known that a weakened form of the Axiom (such as the Denumerable Axiom) suffices to prove P, this fact appears in the text. To say that P is equivalent to the Axiom means that such an equivalence is provable in **ZF**.

It is possible to define the term "implicit use of the Axiom" much more narrowly than we have done. One might regard the Axiom as used implicitly to prove a proposition P if and only if P is equivalent to the Axiom. For historical purposes, such a definition would be too restrictive, since it would exclude pre-1904 uses which mathematicians described soon afterward as implicit. Moreover, such a definition would distort the historical perspective in a second way. For it would strongly suggest that an implicit use of the Axiom was an implicit use of any other proposition P equivalent to the Axiom (such as Tychonoff's Theorem), even though P was only conceived decades later and did not appear in any direct way in that implicit use of the Axiom.

To make an infinity of arbitrary choices the hallmark of an implicit use of the Assumption, as has been done here, does not eliminate all difficulties. Although the Axiom justifies infinitely many arbitrary choices so long as they are independent of each other, mathematicians sometimes made an infinity of arbitrary selections such that a given choice depended on those previously made. This was the case, for example, in the early attempts to well-order an infinite set M by picking one element after another from M (see 1.5). Shortly after 1904, some mathematicians were satisfied with the Axiom partly because it avoided infinitely many such dependent choices. Later it was recognized that the Axiom could justify dependent choices as well.

Most commonly, dependent choices occurred when a mathematician selected a sequence $a_1, a_2, \ldots$ such that the choice of a_{n+1} depended on a_n. In general, such choices cannot be made by using the Denumerable Axiom, even on the real line.[3] However, the existence of such a sequence can be justified by a proposition which follows from the Axiom of Choice and which Paul Bernays proposed in 1942 as a weakened form of the Axiom useful in analysis. This form is now known as the Principle of Dependent Choices:

(1.1.2) If S is a relation on a set A such that for every x in A there exists some y in A with xSy, then there is a sequence $a_1, a_2, \ldots$ such that for every positive integer n, a_n is in A and a_nSa_{n+1} holds.[4]

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[3] Jensen 1966, 294.

[4] Bernays 1942, 86. The name of this principle is due to Tarski [1948, 96]. Azriel Levy [1964, 136] generalized it from sequences of type ω to sequences of type α for any infinite ordinal α.

Whenever a mathematician obtained a sequence prior to 1904 by making such arbitrary dependent choices, it will be termed an implicit use of the Dependent Assumption.

To derive propositions which follow from the Axiom but are weaker than it, one might seek an alternative assumption. Such an alternative could be a restricted form of the Axiom, say the Denumerable Axiom or the Principle of Dependent Choices, but might be another type of proposition altogether. The serious investigation of such alternatives, which began in 1918, did not attract much attention at the time except from the Axiom's Italian critics (see 4.7). A much later and more promising alternative was the Axiom of Determinateness (see the Epilogue).

On the other hand, one must not overlook the radical attempt by L. E. J. Brouwer and his followers, beginning in 1907, to reformulate all of mathematics in intuitionistic terms.[5] Roused by a desire to prune Cantor's infinite sets, the intuitionists formulated a cohesive ideology within a more general constructivist framework. Such constructivists were suspicious of the actual infinite (or at least of Cantor's uncountable cardinals), of existence proofs which did not exhibit a uniquely defined object, and of the Principle of the Excluded Middle as applied to infinite sets. The twentieth century has produced a cornucopia of constructivist approaches to mathematics, each of which requires that one abandon a substantial portion of Cantor's results. Although it would take us too far afield to analyze such approaches in general, constructivist opposition to the Axiom will emerge as a central theme (see 2.3–2.4, 2.7–2.9, 3.6, and 4.11).

Within the Cantorian tradition, one can view Zermelo's axiomatization as answering the question: What is a set? This question has served as a theme in the development of set theory, but one not often discussed openly. Nevertheless, it has been implicit in all attempts since 1908 to modify Zermelo's axioms, or to replace them with others (see 4.9), and in the polemics of his French constructivist critics (see 2.3 and 2.4). In 1913 Michele Cipolla inquired whether the Axiom restricted the general concept of set too greatly [1913, 2]. While no one pursued the matter further at the time, this perspective contrasted sharply with other constructivist views that the Axiom was either false or meaningless.

Since the Axiom appears subtly in many proofs and since it is easily overlooked, we now examine some fundamental theorems for which arguments were given before Zermelo that involved its implicit and unavoidable use. Prior to 1904 such implicit uses occurred in real analysis, algebraic number theory, point-set topology, and set theory.[6] Afterwards the controversy

[5] See for example Kleene and Vesley 1965, Heyting 1966, Bishop 1967, and Gauthier 1977.

[6] In the mathematical literature the term *point-set topology* is ambiguous. Within this book it refers to the theory, sometimes called *Mannigfaltigkeitslehre* or the theory of point-sets, that Cantor originated and that utilized such notions as limit point, closed set, and derived set in $\mathbb{R}^n$. Largely through the work of Hausdorff, this theory later grew into general topology, which considered more general spaces.

surrounding the Axiom led to research, which continues today, as to whether a theorem requires the Axiom and in how strong a form. Thus the Axiom increased awareness of what one may call, in an informal sense, degrees of non-constructivity. Such research would not have begun, had not many mathematicians regarded the Axiom as a dubious assumption.

The first proposition to be considered, hereafter termed the Countable Union Theorem, was employed in both set theory and analysis, beginning with Cantor's researches in the 1870s:

(1.1.3) The union of a countable family of countable sets is countable.

To understand the Axiom's role in demonstrating the Countable Union Theorem, suppose first that each of the sets $A_1, A_2, \ldots$ is denumerable. It follows that each A_i has as its members $a_{i,1}, a_{i,2}, \ldots$. Thus the union B of all the A_i consists of the elements $a_{i,j}$ where i and j are positive integers. Clearly B is denumerable by Cantor's argument showing the rational numbers to be denumerable. Hence a countable union of countable sets is a subset of a denumerable set, and consequently is countable.

Although the Axiom may not be visible here at first, it entered when we enumerated all the members of all the A_i. There are infinitely many A_i, for each of which there exist many possible bijections onto the set of positive integers.[7] By the Denumerable Axiom we associate with each A_i a unique such bijection a_i. Hence the $a_i(j)$, or $a_{i,j}$, are well-defined. This use of the Denumerable Axiom is unavoidable. Indeed, there exists a model of **ZF** in which the Denumerable Axiom is false and the set $\mathbb{R}$ of all real numbers, though uncountable, is a countable union of countable sets.[8]

Our second example concerns the border between the finite and the infinite, a border that can be very nebulous in the absence of the Axiom:

(1.1.4) Every infinite set A has a denumerable subset.[9]

Cantor in 1895, Borel in 1898, and Russell in 1902 all demonstrated this theorem by using the Denumerable Assumption implicitly.[10] Russell's proof illustrates how the Axiom is involved. Since A is infinite, there exist subsets $A_1, A_2, \ldots$ of A such that, for each n, A_n has exactly n members and is a subset of A_{n+1}. The desired denumerable subset of A is the union of all A_n. Apparently Russell did not notice at the time that to form A_{n+1} once A_n has been obtained, one must select some member of $A - A_n$. Since denumerably many such

[7] A *bijection* from a set A to a set B is a one–one function from A onto B.

[8] Feferman and Levy 1963; Cohen 1966, 143–146.

[9] A set A is *finite* if A is empty or if, for some positive integer n, there is a bijection from A onto $\{1, 2, \ldots, n\}$; otherwise A is *infinite*. For a discussion of the Axiom's role in showing various definitions of finite set to be equivalent, see 1.3 and 4.2.

[10] Cantor 1895, 493; Borel 1898, 12–14; Whitehead 1902, 121–123. As Whitehead acknowledged, Russell wrote the section in which this proof appeared.

choices are needed and since no rule is available in general, the proof requires the Denumerable Axiom. For there is a model of **ZF** in which a certain infinite set of real numbers lacks a denumerable subset.[11]

The third example is a proposition, hereafter called the Partition Principle, which appears all but self-evident:

(1.1.5) If a set M is partitioned into a family S of disjoint non-empty sets, then S is equipollent to a subset of M (*i.e.*, $\overline{\overline{S}} \leq \overline{\overline{M}}$).[12]

In terms of a function f with domain M, (1.1.5) states that $\overline{\overline{f``M}} \leq \overline{\overline{M}}$. During the 1880s Cantor employed a special case of the Partition Principle while investigating the topology of the real line.[13] However, the explicit (though incomplete) formulation of the Partition Principle was due to Cesare Burali-Forti [1896, 46]; see 1.3, 1.4, and 1.8. In general, the proof of this principle depends on selecting an element from each set in S so as to obtain a bijection from S onto a subset of M. Thus the proof relies on the Axiom. Moreover, there exists a model of **ZF** in which the Partition Principle is false.[14] At present it is not known whether the Partition Principle is weaker than the Axiom or equivalent to it.[15]

Lastly, we consider the Trichotomy of Cardinals, a theorem closely related to the proposition that every set can be well-ordered:

(1.1.6) For every cardinal $\mathfrak{m}$ and $\mathfrak{n}$, either $\mathfrak{m} < \mathfrak{n}$ or $\mathfrak{m} = \mathfrak{n}$ or $\mathfrak{m} > \mathfrak{n}$.[16]

When formulated in terms of sets rather than cardinals, (1.1.6) states that any two sets A and B are comparable, *i.e.*, one of them is equipollent to a subset of the other. In 1895 Cantor asserted the Trichotomy of Cardinals without proof. Some four years later he wrote to Dedekind that (1.1.6) followed from the proposition that every set can be well-ordered.[17] Their equivalence remained unproven until Friedrich Hartogs established it in 1915.

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[11] Cohen 1966, 138. Neither (1.1.3) nor (1.1.4) implies the other in **ZF**. Sageev [1975] has shown that there is a proposition, namely (1.7.10), which yields (1.1.4) but not (1.1.3) in **ZF**. On the other hand, in a personal communication to the author, Jech observed that (1.1.3) but not (1.1.4) holds in what he called the basic Cohen model; *cf.* Jech 1973, 66, 81.

[12] Two sets are *equipollent* if there is a bijection from one of them onto the other.

[13] Cantor 1883b, 413–414; 1884, 464. Now $f``M$ is $\{f(x): x \in M\}$, *i.e.*, the *image* of M under f.

[14] Jech and Sochor 1966, 352.

[15] Certainly the Partition Principle implies many special cases of the Axiom. Sierpiński [1947b, 157] established that the existence of a non-measurable set follows from this principle, while Andrzej Pelc [1978, 587–588] showed—using work by David Pincus—that it yields both the Principle of Dependent Choices and the Axiom restricted to well-ordered families of sets.

[16] $\mathfrak{m} < \mathfrak{n}$ if, for every set A and B of power $\mathfrak{m}$ and $\mathfrak{n}$ respectively, A is equipollent to a subset of B but not to B itself.

[17] Letters of 28 July and 3 August 1899 in Cantor 1932, 443–447.

These four themes—the Countable Union Theorem, the Partition Principle, the border between the finite and the infinite, and finally the intertwined problems of the Well-Ordering Principle and the Trichotomy of Cardinals—form the warp upon which the woof of the Axiom and its early history will be woven. On the other hand, as the next section illustrates, the first two unavoidable implicit uses of the Axiom arose from quite different sources.

1.2 The Origins of the Assumption

After such preliminary remarks, we can indicate the major stages through which the use of arbitrary choices passed on the way to Zermelo's explicit formulation of the Axiom. In particular the outlines of four stages, though not always their precise historical boundaries, are visible. Vestiges of the first stage—choosing an unspecified element from a single set—can be found in Euclid's *Elements*, if not earlier. Such choices formed the basis for the ancient method of proving a generalization by considering an arbitrary but definite object, and then executing the argument for that object. This first stage also included the arbitrary choice of an element from each of finitely many sets. It is important to understand that the Axiom was not needed for an arbitrary choice from a single set, even if the set contained infinitely many elements. For in a formal system a single arbitrary choice can be eliminated through the use of universal generalization or similar rules of inference. By induction on the natural numbers, such a procedure can be extended to any finite family of sets.

The second stage began when a mathematician made an infinite number of choices by stating a rule. Since the second stage presupposed the existence of an infinite family of sets, two promising candidates for its emergence are nineteenth-century analysis and number theory. In the first case there were analysts who arbitrarily chose the terms of an infinite sequence, and, in the second, number-theorists who selected representatives from infinitely many equivalence classes. When some mathematician, perhaps Cauchy, made such an infinity of choices but left the rule unstated, he initiated the third stage.

This oversight—failing to provide a rule for the selection of infinitely many elements—encouraged the fourth stage to emerge. Thus in 1871, as we shall soon describe, Cantor made an infinite sequence of arbitrary choices for which *no* rule was possible, and consequently the Denumerable Axiom was required for the first time. Nevertheless, Cantor did not recognize the impossibility of specifying such a rule, nor did he understand the watershed which he had crossed. After that date, analysts and algebraists increasingly used such arbitrary choices without remarking that an important but hidden assumption was involved. From this fourth stage emerged Zermelo's solution to the

Well-Ordering Problem and his explicit formulation of the Axiom of Choice.

However, during the early years of the nineteenth century mathematics remained very much, as it had been for Euclid, a process of construction. If one wished to prove that a particular type of mathematical object existed, then one had to construct such an object from those previously shown to exist. On the other hand, since the techniques allowed in such a construction were not precisely delimited, the door was open for infinitely many arbitrary choices to enter unnoticed.

These opposite tendencies are visible in the most significant work on number theory to appear during that period, the *Disquisitiones Arithmeticae* which Gauss published in 1801. While discussing binary quadratic forms $ax^2 + 2bxy + cy^2$, Gauss showed that for those forms with a given discriminant $d = b^2 - ac$ there existed a unique integer n such that they could be partitioned into n classes by means of a certain equivalence relation.[1] Since he recognized that there were many ways to select a representative from each equivalence class, he carefully supplied a rule which determined those representatives uniquely. By choosing infinitely many representatives through a rule, though only a finite number of them for each value of d, Gauss paused on the border between the first and second stages.[2] His unflinchingly algorithmic approach to number theory made it unlikely that he ever entered the third stage.

In all probability the third stage, where infinitely many choices were made by an unstated rule, originated in analysis rather than in number theory. This stage was already evident in 1821 when Cauchy demonstrated a version of the Intermediate Value Theorem:

(1.2.1) Any real function f continuous on the closed interval $[a, b]$ has a root there, provided that $f(a)$ and $f(b)$ have opposite signs.

For a given integer m greater than one, Cauchy noted that the finite sequence

$$f(a),\ f\left(a + \frac{b-a}{m}\right),\ f\left(a + \frac{2(b-a)}{m}\right),\ \ldots,\ f(b)$$

must contain some consecutive pair with opposite signs. He let $f(a_1), f(b_1)$ be one such pair with $a_1 < b_1$ so that $b_1 - a_1 = (b-a)/m$. Next he considered the sequence of points dividing $[a_1, b_1]$ into m equal parts and, as

[1] Gauss 1801, section 223; translated in Gauss 1966. He defined this relation as follows: Two forms A and B are equivalent if they have the same discriminant (he used the term "determinant") and if there is a linear transformation with integer coefficients taking A to B and another such transformation taking B to A.

[2] Gauss [1832, section 42] also stated a rule to determine representatives of equivalence classes when he treated biquadratic residues. Medvedev [1965, 17–18, 23] has erroneously stated that Gauss used the Axiom of Choice implicitly in these works of 1801 and 1832.

Here a conceptual shift from the algorithmic toward the non-constructive was under way.

The fourth stage, where a mathematician made infinitely many arbitrary choices for which no rule was possible and for which, consequently, the Axiom of Choice was essential, began by October 1871 when Eduard Heine wrote an article on real analysis. Printed the following year, the article was largely based on unpublished research by Weierstrass. In all probability Heine learned of Weierstrass's work from Georg Cantor, who had studied under Weierstrass at Berlin and who became Heine's colleague at the University of Halle in 1869. Of the theorems in Heine's article, the one involving arbitrary choices was credited to Cantor:

(1.2.2) A real function f is continuous at a point p if and only if f is sequentially continuous at p. [Heine 1872, 183].

In effect, Cantor's theorem stated that two characterizations of continuity are equivalent. The first was the usual definition in real analysis, due to Cauchy and Weierstrass: A real function f is continuous at a point p if for every $\varepsilon > 0$ there is some $\eta > 0$ such that for every x,

$$|x - p| < \eta \quad \text{implies } |f(x) - f(p)| < \varepsilon.$$

The second characterization, by means of sequences rather than intervals, was what will hereafter be termed sequential continuity: A real function f is sequentially continuous at p if, for every sequence $x_1, x_2, \ldots$ converging to p, the sequence $f(x_1), f(x_2), \ldots$ converges to $f(p)$.[8]

Heine's proof, borrowed from Cantor, implicitly used the Assumption to show that sequential continuity at p yielded continuity there: Suppose, Heine began, that f is not continuous at p. Then there is some positive ε such that no matter how small η_0 is, there is always some positive η less than η_0 such that $|f(p + \eta) - f(p)| \geq \varepsilon$:

> So for any one value of η_0, let one such value of η (smaller than this η_0), for which the above difference $[|f(p + \eta) - f(p)|]$ is not smaller than ε be equal to η'. For half as large a value of η_0 the difference for $\eta = \eta''$ cannot be smaller than ε; for an η_0 equal to half the earlier (a quarter of the first) this must occur for $\eta = \eta'''$, and so on. [1872, 183]

Since the sequence $\eta', \eta'', \ldots$ converges to zero, then $p + \eta', p + \eta'', \ldots$ converges to p; but $f(p + \eta'), f(p + \eta''), \ldots$ does not converge to $f(p)$, contrary to hypothesis.

Neither Cantor nor Heine gave any indication of suspecting that a new and fundamental assumption was required for this proof. Not until a decade

[8] We use the name *sequential continuity* for this property to emphasize how it parallels the definition of continuity while relying on sequences. This notion has also been termed Heine continuity (see Steinhaus 1965, 457).