RING EXAMPLES

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Let $(R, +, \times)$ be a ring. We know that $(R, +)$ is always a group, but a question is "how close is $(R, \times)$ to being a group?" or "an abelian group?" By homework problem 1 from homework 2, we know that 0 will cause problems, so let's look at $R^* = R \setminus \{0\}$. To look at how close $(R^*, \times)$ is to being a group or abelian group, we need to check associativity, closure, the existence of an identity, the existance of inverses, and if the multiplication is commutative. Associativity follows directly from the ring axioms and so the remaining ones are interesting. Closure must be checked because even though there is a ring axiom that says the product of two elements of the ring is also in the ring, it could happen that the product of two elements in R^* is not in R^*. On the next page, there is a chart for you to fill in, if you want to try it out yourself. I will include the complete table at the end.

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	Identity	Inverses	No Zero Divisors	Commutative
Ring				
Commutative Ring				
Commutative Ring with Unity				
Integral Domain				
Division Ring				
Field				
$(\mathbb{Z}, +, \times)$				
$(2\mathbb{Z}, +, \times)$				
$(\mathbb{Q}, +, \times)$				
$(M(2, \mathbb{R}), +, \times)$, 2×2 matrices over $\mathbb{R}$				
$(\mathbb{R}, +, \times)$				
$(\mathbb{Z}_6, +, \times)$				
$(\mathbb{Z}_5, +, \times)$				
$(\mathbb{C}, +, \times)$				
$(Q, +, \times)$, the (real) quaternions				

The rest of these notes will be devoted to filling in this chart. The first few lines of this chart are fairly easy, because they are done axiomatically. Also, note that any ring with the properties that appear in the first few is that type of ring.

1. Ring

None of the properties that we're looking for are guaranteed by the ring axioms. In addition, in the examples below, we will find different rings where each of the properties do not hold. Therefore, for a general ring, none of the properties hold.

2. Commutative Ring

The property that defines a commutative ring is that for all $a, b \in R$, $a \times b = b \times a$. This is exactly the property of commutativity that we were looking for. In the examples below, we will find different commutative rings where the other properties do not hold.

3. Commutative Ring with Unity

A commutative ring with unity is a commutative ring with an element such that $a \times 1 = 1 \times a = a$ for all $a \in R$. Therefore, a commutative ring with unity has the commutative property because it is a commutative ring. It also has an identity because the property of 1 is exactly the property that shows that 1 is the identity. We will have examples below where these are the only properties that hold.

4. Integral Domain

An integral domain is a commutative ring with no zero divisors. Therefore, an integral domain is has the commutative property because it is a commutative ring. It also has no zero divisors because it is defined to not have any zero divisors. We will see examples below that show that there are integral domains with only these two properties.

5. Division Ring

A division ring is defined to be a ring where R^* is a group under multiplication. For this to be a group, it is necessary that R has an identity, inverses, and no zero divisors. The only hard one to see is that there are no zero divisors. If there were zero divisors, then there would be $a, b \in R^*$ such that $ab = 0 \notin R^*$. We will see below a division ring that is not commutative

6. Field

A field is commutative division ring. Therefore a field has all the properties of a division ring and so has an identity, inverses, and no zero divisors. In addition, it is a commutative ring and so has the commutative property.

7. $(\mathbb{Z}, +, \times)$

First, this ring has an identity since $1 \times n = n = n \times 1$ for any real number n and in particular for integers. In addition, for all real numbers $a \times b = b \times a$, in particular for integers, this holds and this is a commutative ring. In addition, for two real numbers if $a \times b = 0$, then either $a = 0$ or $b = 0$, in particular for integers, no two nonzero integers can be multiplied to get zero. Therefore there are no zero divisors in the integers. The tntegers do not have inverses because 2 is an integer, but its multiplicative inverse $1/2$ is not an integer. Therefore, $\mathbb{Z}$ is an integral domain, but not a field. It also is a commutative ring with unity.

8. $(2\mathbb{Z}, +, \times)$

This ring does not have an identity since 1 is not in $2\mathbb{Z}$. The ring is commutative since, as above, for real numbers $a \times b = b \times a$, which holds for integers in particular. It also has no zero divisors because for real numbers $a \times b = 0$, then either $a = 0$ or $b = 0$, thus for integers, this also holds. It cannot have inverses because it does not have an identity. Therefore $2\mathbb{Z}$ is an integral domain.

9. $(\mathbb{Q}, +, \times)$

This ring has an identity since $1 \in \mathbb{Q}$ and $1 \times a = a = a \times 1$ for any real number a and in particular it holds for rational numbers. Also, this ring has inverses, for a non zero element of this ring is of the form p/q where neither p nor q is 0. The inverse of p/q is q/p, we only must show that this is in $\mathbb{Q}$. q/p is in $\mathbb{Q}$ since $p \neq 0$. This ring does not have zero divisors since for real numbers, if $a \times b = 0$, then either $a = 0$ or $b = 0$, thus for rationals, this also holds. Finally, it is commutative since for real numbers $a \times b = b \times a$, which holds for rationals in particular. Therefore $\mathbb{Q}$ is a field.

10. $(M(2, \mathbb{R}), +, \times)$

This ring has an identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ because for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{R})$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This ring has zero divisors since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are not the zero matrix. It is not commutative since $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Finally, this ring does not have inverses, because assume that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ had an inverse A. Then, if we multiply the expression for zero divisors on the left by A, we would have $A \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = A \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $A \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and so the original expression is also $A \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ cannot have an inverse. Therefore $M(2, \mathbb{R})$ is a ring.

11. $(\mathbb{R}, +, \times)$

This ring has an identity since $1 \in \mathbb{R}$ and $1 \times a = a = a \times 1$ for any real number a. Also, this ring has inverses since if $a \neq 0$, then $1/a \in \mathbb{R}$, by construction, and $1/a$ is the multiplicative inverse of a. This ring does not have zero divisors because if $a \times b = 0$, then $a = 0$ or $b = 0$. Finally, this ring is commutative since for reals, $a \times b = b \times a$. Therefore $\mathbb{R}$ is a field.

12. $(\mathbb{Z}_6, +, \times)$

This ring has an identity since $[1] \in \mathbb{Z}_6$ and $[1] \times [a] = [1 \times a] = [a]$ since 1 is the identity in $\mathbb{Z}$. Also, this ring is commutative since $[a] \times [b] = [a \times b] = [b \times a] = [b] \times [a]$, where $a \times b = b \times a$ since $\mathbb{Z}$ is a commutative ring. However, this ring has zero divisors since $[2] \times [3] = [2 \times 3] = [6] = [0]$. Also, this ring does not have inverses, which we can see by inspection. $[1] \times [2] = [2]$, $[2] \times [2] = [4]$, $[3] \times [2] = [0]$, $[4] \times [2] = [2]$, and $[5] \times [2] = [4]$, none of which are equal to $[1]$. Therefore $\mathbb{Z}_6$ is a commutative ring with unity.

13. $(\mathbb{Z}_5, +, \times)$

This ring has an identity since $[1] \in \mathbb{Z}_6$ and $[1] \times [a] = [1 \times a] = [a]$ since 1 is the identity in $\mathbb{Z}$. Also, this ring is commutative since $[a] \times [b] = [a \times b] = [b \times a] = [b] \times [a]$, where $a \times b = b \times a$ since $\mathbb{Z}$ is a commutative ring. This ring does not have zero divisors since if $[a] \times [b] = [a \times b] = [0]$, then 5 divides $a \times b$. Since 5 is prime, is must either divide a or b, in the first case $[a] = [0]$ and in the second case $[b] = [0]$. Therefore, if a product of elements is $[0]$, then one of them must be $[0]$. Finally, this ring has inverses since $[1] \times [1] = [1]$, $[2] \times [3] = [6] = [1]$, $[3] \times [2] = [6] = [1]$, and $[4] \times [4] = [16] = [1]$. Therefore $\mathbb{Z}_5$ is a field.

14. $(\mathbb{C}, +, \times)$

This ring has an identity since $1 \in \mathbb{C}$ and $1 \times a = a = a \times 1$ for any complex number a. Also, this ring has inverses since if $a \neq 0$, then $1/a \in \mathbb{R}$, by construction, and $1/a$ is the multiplicative inverse of a. This ring does not have zero divisors because if $a \times b = 0$, then $a = 0$ or $b = 0$. Finally, this ring is commutative since for complexes, $a \times b = b \times a$. Therefore $\mathbb{C}$ is a field.

15. $(Q, +, \times)$

The real quaterions are formed in the following way: start with a vector space over $\mathbb{R}$, with basis vectors $1, i, j, k$. In other words, an element of Q looks like $a_1 + a_2 i + a_3 j + a_4 k$. Products are determined by the relationships that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, and $ki = j$. Therefore, the product of two elements is $(a_1 + a_2 i + a_3 j + a_4 k) \times (b_1 + b_2 i + b_3 j + b_4 k) = (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4) + (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)i + (a_1 b_3 + a_3 b_1 + a_4 b_2 - a_2 b_4)j + (a_1 b_4 + a_4 b_1 + a_2 b_3 - a_3 b_2)k$. This ring takes a little getting used to so, it's good to experiment with it. This ring has an identity since 1 is the identity. Choosing $a_1 = 1$, $a_2 = 0$, $a_3 = 0$, and $a_4 = 0$, the above product simplifies to $b_1 + b_2 i + b_3 j + b_4 k$. It is not commutative since $ij = k$, but $ji = ji(j)(-j) = -j(ij)j = -jkj = -ij = -k \neq k$, in this, we used that $j(-j) = -j^2 = 1$. Elements have inverses, and the inverse of $a_1 + a_2 i + a_3 j + a_4 k$ is $\frac{a_1}{a_1^2 + a_2^2 + a_3^2 + a_4^2} + \frac{-a_2}{a_1^2 + a_2^2 + a_3^2 + a_4^2} i + \frac{-a_3}{a_1^2 + a_2^2 + a_3^2 + a_4^2} j + \frac{-a_4}{a_1^2 + a_2^2 + a_3^2 + a_4^2} k$, if you plug this into the formula above, you will find that the product is 1. Finally, there are no zero divisors, because if there were, then by the same argument as in the $M(2, \mathbb{R})$ case, we would have a contradiction to the existence of inverses. Therefore, Q is a division ring.

Finally, we are ready combine all of these facts into the table:

	Identity	Inverses	No Zero Divisors	Commutative
Ring				
Commutative Ring				$\times$
Commutative Ring with Unity	$\times$			$\times$
Integral Domain			$\times$	$\times$
Division Ring	$\times$	$\times$	$\times$	
Field	$\times$	$\times$	$\times$	$\times$
$(\mathbb{Z}, +, \times)$	$\times$		$\times$	$\times$
$(2\mathbb{Z}, +, \times)$			$\times$	$\times$
$(\mathbb{Q}, +, \times)$	$\times$	$\times$	$\times$	$\times$
$(M(2, \mathbb{R}), +, \times)$, 2×2 matrices over $\mathbb{R}$	$\times$			
$(\mathbb{R}, +, \times)$	$\times$	$\times$	$\times$	$\times$
$(\mathbb{Z}_6, +, \times)$	$\times$			$\times$
$(\mathbb{Z}_5, +, \times)$	$\times$	$\times$	$\times$	$\times$
$(\mathbb{C}, +, \times)$	$\times$	$\times$	$\times$	$\times$
$(Q, +, \times)$, the (real) quaternions	$\times$	$\times$	$\times$	