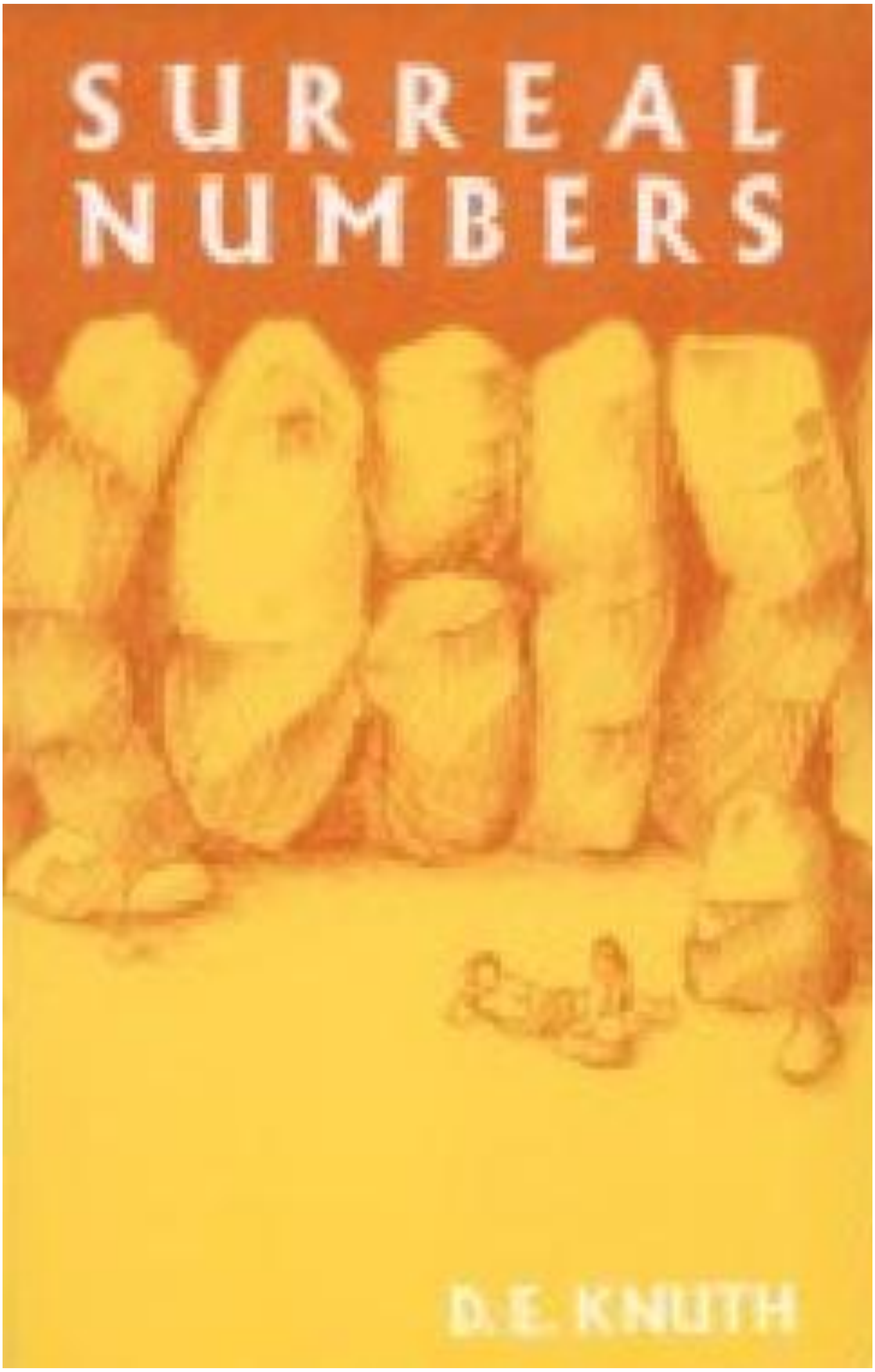

SURREAL
NUMBERS
D.E. KNUTH

7 DISCOVERY

A. Well, we've solved everything on that rock, but I can't help feeling there's still a lot missing.

B. What do you mean?

A. I mean, like we know what happened on the third day, four new numbers were created. But we don't know what Conway called them.

B. Well, one of the numbers was bigger than 1, so I suppose he called it "2." And another was between 0 and 1, so maybe he called it "$\frac{1}{2}$."

A. That's not really the point; what really bothers me is, why are they *numbers*? I mean, in order to be numbers you have to be added, subtracted, and that sort of thing.

B. (frowning) I see. You think Conway gave some more rules, in the broken-off part of the rock, which made the numbers numerical. All we have is a bunch of objects ordered neatly in a line, but we haven't got anything to do with them.

A. I don't think I'm clairvoyant enough to guess what he did—if he did do something.

B. That means we're stuck, unless we can find the missing part of that rock. And I don't remember where we found the first part.

A. Oh, I remember that, I was careful to note exactly where it was in case we ever wanted to go back.

B. What would I do without you? Come on, let's go!

A. Hey wait, don't you think we should have a little lunch first?

B. Right, I got so wrapped up in this I forgot all about food. Okay, let's grab a quick bite and then start digging.

. . . .

A. (digging) Oh, Bill, I'm afraid this isn't going to work. The dirt under the sand is so hard, we need special tools.

B. Yeah, just scraping away with this knife isn't getting us very far. Uh oh, here comes the rain, too. Should we dash back to camp?

A. Look, there's a cave over by that cliff. Let's wait out the storm in there. Hey, it's really pouring!

. . . .

B. Sure is dark in here. Ouch! I stubbed my toe on something. Of all the . . .

A. Bill! You've found it! You stubbed your toe on the other part of the Conway Stone!

B. (wincing) Migosh, it look's like you're right. Talk about fate! But my toe isn't as pleased about it as the rest of me is.

A. Can you read it, Bill? Is it really the piece we want, or is it something else entirely?

B. It's too dark in here to see much. Help me drag it out in the rain, the water will wash the dust off and . . .

Yup, I can make out the words "Conway" and "number," so it *must* be what we're looking for.

A. Oh, good, we'll have plenty to work on. We're saved!

B. The info we need is here all right. But I'm going back in the cave, it can't keep raining this hard for very long.

A. (following) Right, we're getting drenched.

. . . .

B. I wonder why this mathematics is so exciting now, when it was so dull in school. Do you remember old Professor Landau's lectures? I used to really hate that class: Theorem, proof, lemma, remark, theorem, proof, what a total drag.

A. Yes, I remember having a tough time staying awake. But look—wouldn't *our* beautiful discoveries be just about the same?

B. True. I've got this mad urge to get up before a class and present our results: Theorem, proof, lemma, remark. I'd make it so slick, nobody would be able to guess how we did it, and everyone would be *so* impressed.

A. Or bored.

B. Yes, there's that. I guess the excitement and the beauty comes in the discovery, not the hearing.

A. But it *is* beautiful. And I enjoyed hearing your discoveries almost as much as making my own. So what's the real difference?

B. I guess you're right, at that. I was able to really appreciate what *you* did, because I had already been struggling with the same problem myself.

A. It was dull before, because we weren't involved at all; we were just being told to absorb what somebody else did, and for all we knew there was nothing special about it.

B. From now on whenever I read a math book, I'm going to try to figure out by myself how everything was done, before looking at the solution. Even if I don't figure it out, I think I'll be able to see the beauty of a proof then.

A. And I think we should also try to guess what theorems are coming up; or at least, to figure out how and why anybody would try to prove such theorems in the first place. We should imagine ourselves in the discoverer's place. The creative part is really more interesting than the deductive part. Instead of concentrating just on finding good answers to questions, it's more important to learn how to find good questions!

B. You've got something there. I wish our teachers would give us problems like, "Find something interesting about x," instead of "Prove x."

A. Exactly. But teachers are so conservative, they'd be afraid of scaring off the "grind" type of students who obediently and mechanically do all the homework. Besides, they wouldn't like the extra work of grading the answers to nondirected questions.

The traditional way is to put off all creative aspects until the last part of graduate school. For seventeen or more years, a student is taught examsmanship, then suddenly after passing enough exams in graduate school he's told to do something original.

B. Right. I doubt if many of the really original students have stuck around that long.

A. Oh, I don't know, maybe they're original enough to find a way to enjoy the system. Like putting themselves into the subject, as we were saying. That would make the traditional college courses tolerable, maybe even fun.

B. You always were an optimist. I'm afraid you're painting too rosy a picture. But look, the rain has stopped, let's lug this rock back to camp and see what it says.

8 ADDITION

A. The two pieces fit pretty well, it looks like we've got almost the whole message. What does it say?

B. This part is a little harder to figure out, there are some obscure words, but I think it goes like this:

> **. . . day. And Conway said, "Let the numbers be added to each other in this wise: The left set of the sum of two numbers shall be the sums of all left parts of each number**

with the other; and in like manner the right set shall be from the right parts, each according to his kind." Conway proved that every number plus zero is unchanged, and he saw that addition was good. And the evening and the morning were the third day.

And Conway said, "Let the negative of a number have as its sets the negatives of the number's opposite sets; and let subtraction be addition of the negative." And it was so. Conway proved that subtraction was the inverse of addition, and this was very good. And the evening and the morning were the fourth day.

And Conway said to the numbers, "Be fruitful and multiply. Let part of one number be multiplied by another and added to the product of the first number by part of the other, and let the product of the parts be subtracted. This shall be done in all possible ways, yielding a number in the left set of the product when the parts are of the same kind, but in the right set when they are of opposite kinds." Conway proved that every number times one is unchanged. And the evening and the morning were the fifth day.

And behold! When the numbers had been created for infinitely many days, the universe itself appeared. And the evening and the morning were ℵ day.

And Conway looked over all the rules he had made for numbers, and saw that they were very, very good. And he commanded them to be for signs, and series, and quotients, and roots.

Then there sprang up an infinite number less than infinity. And infinities of days brought forth multiple orders of infinities.

That's the whole bit.

A. What a weird ending. And what do you mean "aleph day"?

B. Well, aleph is a Hebrew letter and it's just standing there by itself, look: $\aleph$. It seems to mean infinity. Let's face it, it's heavy stuff and it's not going to be easy to figure out what this means.

A. Can you write it all down while I fix supper? It's too much for me to keep in my head, and I can't read it.

B. Okay, that'll help me get it clearer in my own mind too.

. . . .

A. It's curious that the four numbers created on the third day aren't mentioned. I still wonder what Conway called them.

B. Maybe if we try the rules for addition and subtraction we could figure out what the numbers are.

A. Yeah, *if* we can figure out those rules for addition and subtraction. Let's see if we can put the addition rule into symbolic form, in order to see what it means . . . I suppose "its own kind" must signify that left goes with left, and right with right. What do you think of this:

$$x + y = ((X_L + y) \cup (Y_L + x), (Y_R + x) \cup (X_R + y)). \tag{3}$$

B. Looks horrible. What does *your* rule mean?

A. To get the left set of $x + y$, you take all numbers of the form $x_L + y$, where x_L is in X_L, and also all numbers $y_L + x$ where y_L is in Y_L. The right set is from the right parts, "in like manner."

B. I see, a "left part" of x is an element of X_L. Your symbolic definition certainly seems consistent with the prose one.

A. And it makes sense too, because each $x_L + y$ and $x + y_L$ ought to be less than $x + y$.

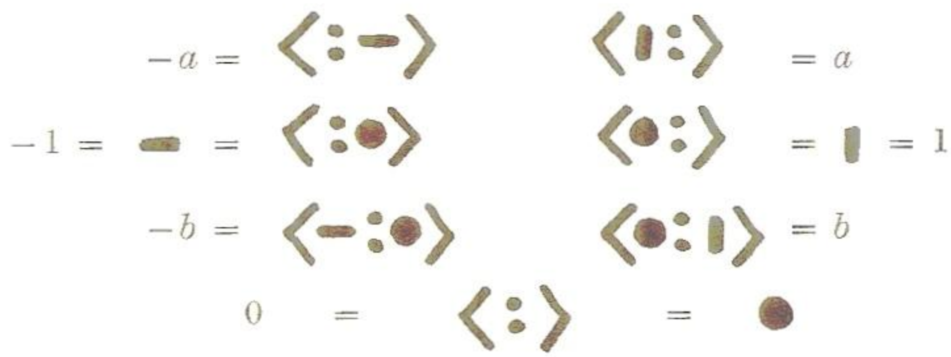
B. Okay, I'm willing to try it and see how it works. I see you've called it rule (3).

A. Now after the third day, we know that there are seven numbers, which we might call 0, 1, -1, a, b, c, and d.

B. No, I have an idea that we can use left-right symmetry and call them

$$-a < -1 < -b < 0 < b < 1 < a,$$

where



A. Brilliant! You must be right, because Conway's next rule is

$$-x = (-X_R, -X_L), \tag{4}$$

B. So it is! Okay—now we can start adding these numbers. Like, what's $1 + 1$, according to rule (3)?

A. You work on that, and I'll work on $1 + a$.

B. Okay, I get $(\{0 + 1, 0 + 1\}, \emptyset)$. And $0 + 1$ is $(\{0 + 0\}, \emptyset)$, $0 + 0$ is $(\emptyset, \emptyset) = 0$. Everything fits together, making $1 + 1 = (\{1\}, \emptyset) = a$. Just as we thought, a must be 2!

A. Congratulations on coming up with the world's longest proof that 1 + 1 is 2.

B. Have you ever seen a shorter proof?

A. Not really. Look, your calculations help me too. I get 1 + 2 = $(\{2\}, \emptyset)$, a number that isn't created until the fourth day.

B. I suggest we call it "3."

A. Bravo. So rule (3) is working; let's check if b is $\frac{1}{2}$ by calculating $b + b$ . . .

B. Hmm, that's odd, it comes out to $(\{b\}, \{b + 1\})$, which hasn't been created yet.

A. And $b + 1$ is $(\{b, 1\}, \{2\})$, which is like $(\{1\}, \{2\})$, which is created on the fourth day. So $b + b$ appears on the *fifth* day.

B. Don't tell me $b + b$ is going to be equal to *another* number we don't know the name of.

A. Are we stuck?

B. We worked out a theory that tells us how to calculate all numbers that are created, so we *should* be able to do this. Let's make a table for the first four days.

A. Oh, Bill, that's too much work.

B. No, it's a simple pattern really. Look:

A. Oh I see, so $b + b$ is $(b, b + 1)$, which is formed from *non-adjacent* numbers . . . And our theory says it is the *earliest-created* number between them.

B. (beaming) And that's 1, because 1 makes the scene before c.

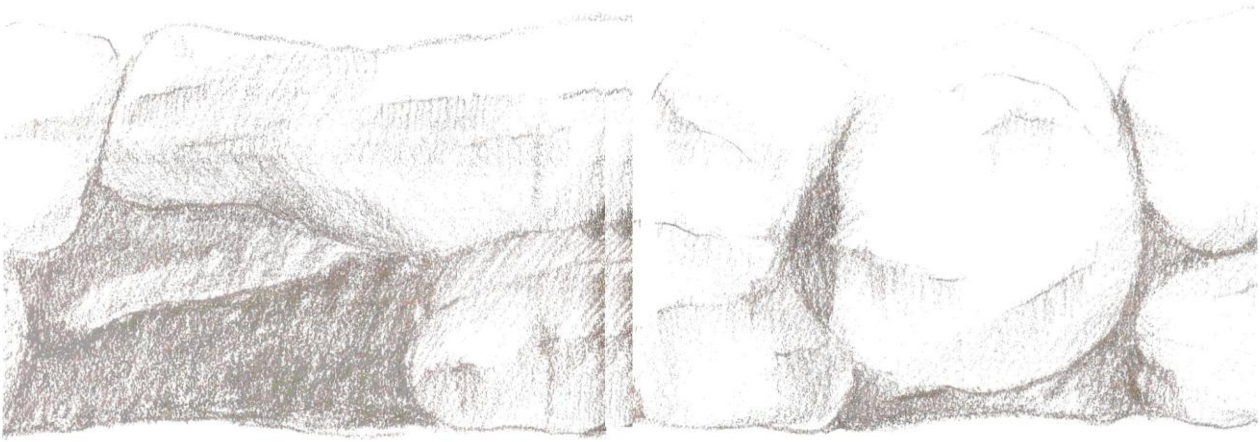
A. So b is $\frac{1}{2}$ after all, although its numerical value wasn't established until two days later. It's amazing what can be proved from those few rules—they all hang together so tightly, it boggles the mind.

B. I'll bet d is $\frac{1}{3}$ and c is $\frac{2}{3}$.

A. But the sun is going down. Let's sleep on it, Bill; we've got lots of time and I'm really drained.

B. (muttering) $d + c =$ . . . Oh, all right. G'night.

9 THE ANSWER



A. Are you awake already?

B. What a miserable night! I kept tossing and turning, and my mind was racing in circles. I dreamed I was proving things and making logical deductions, but when I woke up they were all foolishness.

A. Maybe this mathematics isn't good for us after all. We were so happy yesterday, but—

B. (interrupting) Yeah, yesterday we were high on math, but today it's turning sour. I can't get it out of my system, we've *got* to get more results before I can rest. Where's that pencil?

A. Bill, you need some breakfast. There are some apricots and figs over there.

B. Okay, but I've gotta get right to work.

A. Actually I'm curious to see what happens too, but promise me one thing.

B. What?

A. We'll only work on addition and subtraction today; *not* multiplication. We won't even *look* at that other part of the tablet until later.

B. Agreed. I'm almost willing to postpone the multiplication indefinitely, since it looks awfully complicated.

A. (kissing him) Okay, now relax.

B. (stretching) You're so good to me, Alice.

A. That's better. Now I was thinking last night about how you solved the problem about all the numbers yesterday morning. I think it's an important principle that we ought to write down as a theorem. I mean:

> Given any number y, if x is the first number created with the property that $Y_L < x$ and $x < Y_R$, then $x \equiv y$. (T8)

B. Hmm, I guess that *is* what we proved. Let's see if we can reconstruct the proof, in this new symbolism. As I recall we constructed the number $z = (Y_L \cup X_L, X_R \cup Y_R)$, and then we had $x \equiv z$ by (T7). On the other hand, no element x_L of X_L satisfies $Y_L < x_L$, since x_L was created before x; therefore each x_L is $\leq$ some y_L, by (T4). Thus $X_L < y$, and similarly, $y < X_R$. So $y \equiv z$ by (T7).

It's pretty easy to work out the proof now that we have all this ammunition to work with.

A. The nice thing about (T8) is that it makes the calculation we did last night much easier. Like when we were calculating $b + b = (\{b\}, \{b + 1\})$, we could have seen immediately that 1 is the first number created between $\{b\}$ and $\{b + 1\}$.

B. Hey, let me try that on $c + c$: It's the first number created between $b + c$ and $1 + c$. Well, it must be $b + 1$, I mean $1\frac{1}{2}$, so c is $\frac{3}{4}$.

That's a surprise, I thought it would be $\frac{2}{3}$.

A. And d is $\frac{1}{4}$.

B. Right.

A. I think the general pattern is becoming clear now: After four days the numbers ≥ 0 are

$$0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1, \tfrac{3}{2}, 2, 3$$

and after five days they will probably be—

B. (interrupting)

$$0, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{3}{8}, \tfrac{1}{2}, \tfrac{5}{8}, \tfrac{3}{4}, \tfrac{7}{8}, 1, \tfrac{5}{4}, \tfrac{3}{2}, \tfrac{7}{4}, 2, \tfrac{5}{2}, 3, 4.$$

A. Exactly. Can you prove it?

B. . . .

Yes, but not so easily as I thought. For example, to figure out the value of $f = (\{\frac{3}{2}\}, \{2\})$, which turned out to be $\frac{7}{4}$, I calculated $f + f$. This is the first number created between 3 and 4, and I had to "look ahead" to see that it was $\frac{7}{2}$. I'm convinced we have the right general pattern, but it would be nice to have a proof.

A. On the fourth day we calculated $\frac{3}{2}$ by knowing that it was $1 + \frac{1}{2}$, *not* by trying $\frac{3}{2} + \frac{3}{2}$. Maybe adding 1 will do the trick.

B. Let's see . . . According to the definition, rule (3),

$$1 + x = ((1 + X_L) \cup \{x\}, 1 + X_R),$$

assuming that $0 + x = x$. In fact, isn't it true that . . . sure, for positive numbers we can always choose X_L so that $1 + X_L$ has an element $\geq x$, so it simplifies to

$$1 + x = (1 + X_L, 1 + X_R)$$

in this case.

A. That's it, Bill! Look at the last eight numbers on the fifth day, they are just one greater than the eight numbers on the fourth day.

B. A perfect fit. Now all we have to do is prove the pattern for the numbers x between 0 and 1 . . . but that can always be done by looking at $x + x$, which will be less than 2!

A. Yes, now I'm sure we've got the right pattern.

B. What a load off my mind. I don't even feel the need to formalize the proof now; I *know* it's right.

A. I wonder if our rule for $1 + x$ isn't a special case of a more general rule. Like, isn't

$$y + x = (y + X_L, y + X_R)?$$

That would be much simpler than Conway's complicated rule.

B. Sounds logical, since adding y should "shift" things over by y units. Whoops, no, take $x = 1$; that would say $y + 1$ is $(\{y\}, \emptyset)$, which fails when y is $\frac{1}{2}$.

A. Sorry. In fact, your rule for $1 + x$ doesn't work when $x = 0$ either.

B. Right, I proved it only when x is positive.

A. I think we ought to look at rule (3), the addition rule, more closely and see what can be proved in general from it. All we've got are *names* for the numbers. These names must be correct if Conway's numbers behave like actual numbers, but we don't know that Conway's rules are really the same. Besides, I think it's fun to derive a whole bunch of things from just a few basic rules.

B. Let's see. In the first place, addition is obviously what we might call commutative, I mean

$$x + y = y + x. \tag{T9}$$

A. True. Now let's prove what Conway claimed, that

$$x + 0 = x. \tag{T10}$$

B. The rule says that

$$x + 0 = (X_L + 0, X_R + 0).$$

So all we do is a "day of creation" induction argument, again; we can assume that $X_L + 0$ is the same as X_L, and $X_R + 0$ is X_R, since all those numbers were created before x. Q.E.D.

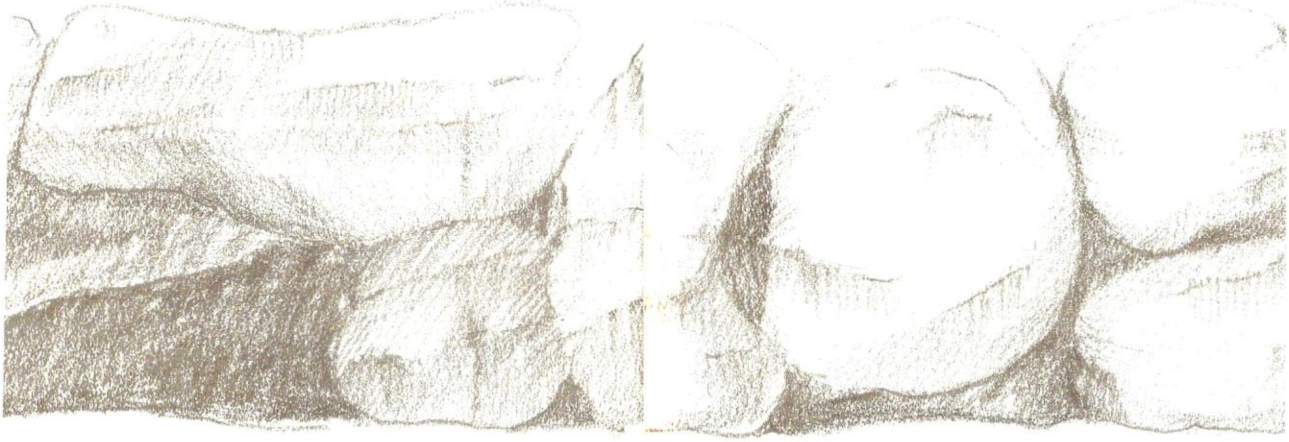
A. Haven't we proved that $x + 0 \equiv x$, not $= x$?

B. You're a nit-picker, you are. I'll change (T10) if you want me to, since it really won't make any difference. But actually doesn't the proof actually show that $x + 0$ is identically the same pair of sets as x?

A. Excuse me again. You're right.

B. That's ten theorems. Shall we try for more while we're hot?

10 THEOREMS



A. How about the associative law,

$$(x + y) + z = x + (y + z). \qquad \text{(T11)}$$

B. Oh, I doubt if we'll need that; it didn't come up in the calculations. But I suppose it won't hurt to try it, since my math teachers always used to think it was such a great thing.

One associative law, coming right up. Can you work out the definition?

A.
$$(x + y) + z = (((X_L + y) + z) \cup ((Y_L + x) + z) \cup (Z_L + (x + y)), ((X_R + y) + z) \cup ((Y_R + x) + z) \cup (Z_R + (x + y)))$$

$$x + (y + z) = ((X_L + (y + z)) \cup ((Y_L + z) + x) \cup ((Z_L + y) + x), (X_R + (y + z)) \cup ((Y_R + z) + x) \cup ((Z_R + y) + x)).$$

B. You're really good at these hairy formulas. But how can such monstrous things be proved equal?

A. It's not hard, just using a day-sum argument on (x, y, z) as we did before. See, $(X_L + y) + z = X_L + (y + z)$ because (x_L, y, z) has a smaller day-sum than (x, y, z), and we can induct on that. The same for the other five sets, using the commutative law in some cases.

B. Congratulations! Another Q.E.D., and another proof of $=$ instead of $\equiv$.

A. That $\equiv$ worries me a little, Bill. We showed that we could substitute like elements for like elements, with respect to $<$ and $\leq$, but don't we have to verify this also for addition? I mean,

$$\text{if} \quad x \equiv y, \quad \text{then} \quad x + z \equiv y + z. \qquad \text{(T12)}$$

B. I suppose so, otherwise we wouldn't strictly be allowed to make the simplifications we've been making in our names for the numbers. As long as we're proving things, we might as well do it right.

A. In fact, we might as well prove a stronger statement,

$$\text{if} \quad x \leq y, \quad \text{then} \quad x + z \leq y + z, \qquad \text{(T13)}$$

because this will immediately prove (T12).

B. I see, because $x \equiv y$ if and only if $x \leq y$ and $y \leq x$. Also (T13) looks like it will be useful. Shouldn't we also prove more, I mean

$$\text{if} \quad x \leq y \quad \text{and} \quad w \leq z,$$
$$\text{then} \quad x + w \leq y + z\,?$$

A. Oh, that follows from (T13), since $x + w \leq y + w = w + y \leq z + y = y + z$.

B. Okay, that's good, because (T13) is simpler. Well, you're the expert on formulas, what is (T13) equivalent to?

A. Given that $X_L < y$ and $x < Y_R$, we must prove that $X_L + z < y + z$, $Z_L + x < y + z$, $x + z < Y_R + z$, and $x + z < Z_R + y$.

B. Another day-sum induction, eh? Really, these are getting too easy.

A. Not quite so easy, this time. I'm afraid the induction will only give us $X_L + z \leq y + z$, and so on; it's conceivable that $x_L < y$ but $x_L + z \equiv y + z$.

B. Oh yeah. That's interesting. What we need is the converse,

$$\text{if} \quad x + z \leq y + z \quad \text{then} \quad x \leq y. \qquad \text{(T14)}$$

A. Brilliant! The converse is equivalent to this: Given that $X_L + z < y + z$, $Z_L + x < y + z$, $x + z < Y_R + z$, and $x + z < Z_R + y$, prove that $X_L < y$ and $x < Y_R$.

B. Hmm. The converse would go through by induction—except that we might have a case with, say, $x_L + z < y + z$ but $x_L \equiv y$. Such cases would be ruled out by (T13), but . . .

A. But we need (T13) to prove (T14), and (T14) to prove (T13). And (T13) to prove (T12).

B. We're going around in circles again.

A. Ah, but there's a way out, we'll prove them *both* together! We can prove the combined statement "(T13) and (T14)" by induction on the day-sum of (x, y, z)!

B. (glowing) Alice, you're a genius! An absolutely gorgeous, tantalizing genius!

A. Not so fast, we've still got work to do. We had better show that

$$x - x \equiv 0. \tag{T15}$$

B. What's that minus sign? We never wrote down Conway's rule for subtraction.

A.
$$x - y = x + (-y). \tag{5}$$

B. I notice you put the $\equiv$ in (T15); okay, it's clear that $x + (-x)$ won't be identically equal to 0, I mean with empty left and right sets, unless x is 0.

A. Rules (3), (4), and (5) say that (T15) is equivalent to this:

$$((X_L + (-x)) \cup ((-X_R) + x),$$
$$(X_R + (-x)) \cup ((-X_L) + x)) \equiv 0.$$

B. Uh oh, it looks hard. How do we show something $\equiv 0$ anyway? . . . By (T8), $y \equiv 0$ if and only if $Y_L < 0$ and $0 < Y_R$, since 0 was the first created number of all.

A. The same statement also follows immediately from rule (2); I mean, $y \leq 0$ if and only if $Y_L < 0$ and $0 \leq y$ if and only if $0 < Y_R$. So now what we have to prove is

$$\begin{array}{rcl} x_L + (-x) < 0, & \text{and} & (-x_R) + x < 0, \\ \text{and } x_R + (-x) > 0, & \text{and} & (-x_L) + x > 0, \end{array}$$

for all x_L in X_L and all x_R in X_R.

B. Hmm. Aren't we allowed to assume that $x_L + (-x_L) \equiv 0$ and $x_R + (-x_R) \equiv 0$?

A. Yes, since we can be proving (T15) by induction.

B. Then I've got it! If $x_L + (-x)$ were ≥ 0, then $(-X)_R + x_L$ would be > 0, by definition. But $(-X)_R$ is $-(X_L)$, which contains $-x_L$, and $(-x_L) + x_L$ is not > 0. Therefore $x_L + (-x)$ must be < 0, and the same technique works for the other cases too.

A. Bravo! That settles (T15).

B. What next?

A. How about this?

$$-(-x) = x. \tag{T16}$$

B. Sssss. That's trivial. Next?

A. All I can think of is Conway's theorem,

$$(x + y) - y \equiv x. \tag{T17}$$

B. What's that equivalent to?

A. It's a real mess . . . Can't we prove things without going back to the definitions each time?

B. Aha! Yes, it almost falls out by itself:

$$\begin{aligned}
(x + y) - y &= (x + y) + (-y) && \text{by (5)}\\
&= x + (y + (-y)) && \text{by (T11)}\\
&= x + (y - y) && \text{by (5)}\\
&= x + 0 && \text{by (T12) and (T15)}\\
&= x. && \text{by (T10)}
\end{aligned}$$

We've built up quite a pile of useful results—even the associative law has come in handy. Thanks for suggesting it against my better judgment.

A. Okay, we've probably exhausted the possibilities of addition, negation, and subtraction. There are some more things we could probably prove, like

$$-(x + y) = (-x) + (-y), \tag{T18}$$

$$\text{if} \quad x \le y, \quad \text{then} \quad -y \le -x, \tag{T19}$$

but I don't think they involve any new ideas; so there's little point in proving them unless we need 'em.

B. Nineteen theorems, from just a few primitive rules.

A. Now you must remember your promise: This afternoon we take a vacation from mathematics, without looking at the rest of the stone again. I don't want that horrible multiplication jazz to rob you of any more sleep.

B. We've done a good day's work, anyhow—all the problems are resolved. Look, the tide's just right again. Okay—the last one into the water has to cook supper!

11 THE PROPOSAL

A. That sure was a good supper you cooked.

B. (lying down beside her) Mostly because of the fresh fish you caught.

What are you thinking about now?

A. (blushing) Well, actually I was wondering what would happen if I got pregnant.

B. You mean, here we are, near the Fertile Crescent, and . . . ?

A. Very funny. And after all our work to prove that $1 + 1 = 2$ we'll discover that $1 + 1 = 3$.

B. Okay, you win, no more jokes. But come to think of it, Conway's rules for numbers are like copulation, I mean the left set meeting the right set, . . .

A. You've got just one thing—no, two things—on your mind. But seriously, what would we do if I really were pregnant?

B. Well, I've been thinking we'd better go back home pretty soon anyway; our money's running out, and the weather is going to get bad.

Actually, I really want to marry you in any case, whether you're pregnant or not. If you'll have me, of course.

A. That's just what I feel too. This trip has proved that we're ready for a permanent relationship.

I wonder . . . When our children grow up, will we teach them our theory of numbers?

B. No, it would be more fun for them to discover it for themselves.

A. But people can't discover *everything* for themselves, there has to be some balance.

B. Well, isn't all learning really a process of self-discovery? Don't the best teachers help their students to think on their own?

A. In a way, yes. Whew, we're getting philosophical.

B. I still can't get over how great I feel when I'm doing this crazy mathematics; it really turns me on right now, but I used to hate it.

A. Yes, I've been high on it, too. I think it's a lot better than drugs; I mean, the brain can stimulate itself naturally.

B. And it was kind of an aphrodisiac, besides.

A. (gazing at the stars) One nice thing about pure mathematics—the things we proved today will never be good for anything, so nobody will be able to use them to make bombs or stuff like that.

B. Right. But we can't be in an ivory tower all the time, either. There are lots of problems in the world, and the right kind of math might help to solve them. You know, we've been away from newspapers for so long, we've forgotten all the problems.

A. Yeah, sometimes I feel guilty about that . . . .

Maybe the right kind of mathematics would help solve some of these problems, but I'm worried that it could also be misused.

B. That's the paradox, and the dilemma. Nothing can be done without tools, but tools can be used for bad things as well as good. If we stop creating things, because they might be harmful in the wrong hands, then we also stop doing useful things.

A. Okay, I grant you that pure mathematics isn't the answer to everything. But are you going to abolish it entirely just because it doesn't solve the world's problems?

B. Oh no, don't misunderstand me. These past few days have shown me that pure mathematics is beautiful—it's an art form like poetry or painting or music, and it turns us on. Our natural curiosity has to be satisfied. It would destroy us if we couldn't have some fun, even in the midst of adversity.

A. Bill, it's good to talk with you like this.

B. I'm enjoying it too. It makes me feel closer to you, and sort of peaceful.