



**Pedro Paiva Zühlke d'Oliveira**

**Homotopies of Curves on the 2-Sphere with  
Geodesic Curvature in a Prescribed Interval**

**Tese de Doutorado**

Thesis presented to the Programa de Pós-Graduação em Matemática of the Departamento de Matemática, PUC-Rio as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Nicolau Corção Saldanha

Rio de Janeiro  
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## Abstract

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For  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ , let  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  denote the set of all closed curves of class  $C^r$  on the sphere  $\mathbf{S}^2$  whose geodesic curvatures lie in the interval  $(\kappa_1, \kappa_2)$ , furnished with the  $C^r$  topology (for some  $r \geq 2$ ). In 1970, J. Little proved that the space  $\mathcal{L}_0^{+\infty}$  of closed curves having positive geodesic curvature has three connected components. Let  $\rho_i = \operatorname{arccot} \kappa_i$  ( $i = 1, 2$ ). In this thesis, we show that  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  has  $n$  connected components  $\mathcal{L}_1, \dots, \mathcal{L}_n$ , where

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

and  $\mathcal{L}_j$  contains circles traversed  $j$  times ( $1 \leq j \leq n$ ). The component  $\mathcal{L}_{n-1}$  also contains circles traversed  $(n-1) + 2k$  times, and  $\mathcal{L}_n$  contains circles traversed  $n + 2k$  times, for any  $k \in \mathbf{N}$ . In addition, each of  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  is homotopy equivalent to  $\mathbf{SO}_3$  ( $n \geq 3$ ). A direct characterization of the components in terms of the properties of a curve and a proof that  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is homeomorphic to  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}$  whenever  $\rho_1 - \rho_2 = \bar{\rho}_1 - \bar{\rho}_2$  ( $\bar{\rho}_i = \operatorname{arccot} \bar{\kappa}_i$ ) are also presented.

## Keywords

Curve. Curvature. Geometry. Homotopy. Topology.

## Resumo

Zühlke, Pedro; Saldanha, Nicolau C.. **Homotopias de Curvas na Esfera com Curvatura Geodésica num Intervalo Dado**. Rio de Janeiro, 2012. 117p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Para  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ , seja  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  o conjunto de todas as curvas fechadas de classe  $C^r$  na esfera  $\mathbf{S}^2$  cujas curvaturas geodésicas estão restritas ao intervalo  $(\kappa_1, \kappa_2)$ , munido da topologia  $C^r$  (para algum  $r \geq 2$ ). Em 1970, J. Little provou que o espaço  $\mathcal{L}_0^{+\infty}$  de curvas fechadas com curvatura geodésica positiva possui três componentes conexas. Sejam  $\rho_i = \operatorname{arccot} \kappa_i$  ( $i = 1, 2$ ). Nesta tese, mostramos que  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  possui  $n$  componentes conexas  $\mathcal{L}_1, \dots, \mathcal{L}_n$ , onde

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

e  $\mathcal{L}_j$  contém círculos percorridos  $j$  vezes ( $1 \leq j \leq n$ ). A componente  $\mathcal{L}_{n-1}$  também contém círculos percorridos  $(n-1) + 2k$  vezes, e  $\mathcal{L}_n$  contém círculos percorridos  $n + 2k$  vezes, para qualquer  $k \in \mathbf{N}$ . Além disto,  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  são todos homotopicamente equivalentes a  $\mathbf{SO}_3$  ( $n \geq 3$ ). Também são exibidas uma caracterização das componentes em termos das propriedades de uma curva e uma prova de que  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  é homeomorfo a  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}$  se  $\rho_1 - \rho_2 = \bar{\rho}_1 - \bar{\rho}_2$  ( $\bar{\rho}_i = \operatorname{arccot} \bar{\kappa}_i$ ).

## Palavras-chave

Curva. Curvatura. Geometria. Homotopia. Topologia.

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