6.252 NONLINEAR PROGRAMMING

LECTURE 11

CONSTRAINED OPTIMIZATION;

LAGRANGE MULTIPLIERS

LECTURE OUTLINE

- Equality Constrained Problems
- Basic Lagrange Multiplier Theorem
- Proof 1: Elimination Approach
- Proof 2: Penalty Approach

Equality constrained problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to } h_i(x) = 0, \qquad i = 1, \ldots, m. \end{aligned}$$

where $f : \Re^n \mapsto \Re$, $h_i : \Re^n \mapsto \Re$, $i = 1, \ldots, m$, are continuously differentiable functions. (Theory also applies to case where f and h_i are cont. differentiable in a neighborhood of a local minimum.)

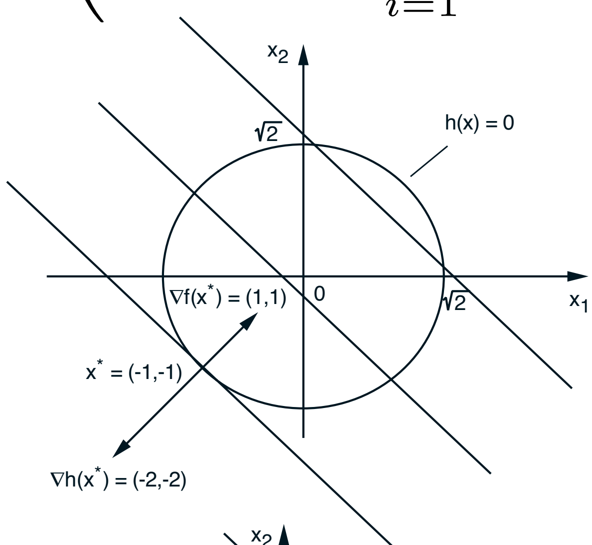
LAGRANGE MULTIPLIER THEOREM

- Let x^* be a local min and a regular point [$\nabla h_i(x^*)$: linearly independent]. Then there exist unique scalars $\lambda_1^*, \ldots, \lambda_m^*$ such that

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) = 0.$$

If in addition f and h are twice cont. differentiable,

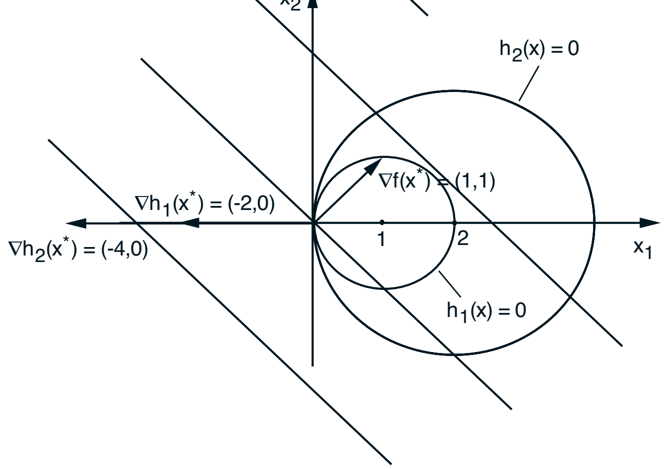
$$y' \left(\nabla^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(x^*) \right) y \geq 0, \ \forall \, y \text{ s.t. } \nabla h(x^*)' y = 0$$



$$\text{minimize} \ \ x_1 + x_2$$

$$\text{subject to} \ \ x_1^2 + x_2^2 = 2.$$

The Lagrange multiplier is $\lambda = 1/2$.



$$\text{minimize} \ \ x_1 + x_2$$

$$\text{s. t. } (x_1 - 1)^2 + x_2^2 - 1 = 0$$

$$(x_1 - 2)^2 + x_2^2 - 4 = 0$$

PROOF VIA ELIMINATION APPROACH

- Consider the linear constraints case

$$\text{minimize } f(x)$$

$$\text{subject to } Ax = b$$

where A is an $m \times n$ matrix with linearly independent rows and $b \in \Re^m$ is a given vector.

- Partition $A = (\,B \quad R\,)$, where B is $m \times m$ invertible, and $x = (\,x_B \quad x_R\,)'$. Equivalent problem:

$$\text{minimize } F(x_R) \equiv f\Big(B^{-1}(b - Rx_R), x_R\Big)$$

$$\text{subject to } x_R \in \Re^{n-m}.$$

- Unconstrained optimality condition:

$$0 = \nabla F(x_R^*) = -R'(B')^{-1}\nabla_B f(x^*) + \nabla_R f(x^*) \quad (1)$$

By defining

$$\lambda^* = -(B')^{-1}\nabla_B f(x^*),$$

we have $\nabla_B f(x^*) + B'\lambda^* = 0$, while Eq. (1) is written $\nabla_R f(x^*) + R'\lambda^* = 0$. Combining:

$$\nabla f(x^*) + A'\lambda^* = 0$$

ELIMINATION APPROACH - CONTINUED

- Second order condition: For all $d \in \Re^{n-m}$

$$0 \leq d' \nabla^2 F(x_R^*) d = d' \nabla^2 \Big(f\Big(B^{-1}(b - Rx_R), x_R \Big) \Big) d. \quad (2)$$

- After calculation we obtain

$$\nabla^2 F(x_R^*) = \ R'(B')^{-1} \nabla^2_{BB} f(x^*) B^{-1} R$$
$$- R'(B')^{-1} \nabla^2_{BR} f(x^*) - \nabla^2_{RB} f(x^*) B^{-1} R + \nabla^2_{RR} f(x^*).$$

- Eq. (2) and the linearity of the constraints [implying that $\nabla^2 h_i(x^*) = 0$], yields for all $d \in \Re^{n-m}$

$$\begin{aligned} 0 \leq d' \nabla^2 F(x_R^*) d &= y' \nabla^2 f(x^*) y \\ &= y' \left(\nabla^2 f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(x^*) \right) y, \end{aligned}$$

where $y = (\, y_B \quad y_R \,)' = (\, -B^{-1} R d \quad d \,)'$.

- y has this form iff

$$0 = B y_B + R y_R = \nabla h(x^*)' y.$$

PROOF VIA PENALTY APPROACH

- Introduce, for $k = 1, 2, \ldots$, the cost function

$$F^k(x) = f(x) + \frac{k}{2}||h(x)||^2 + \frac{\alpha}{2}||x - x^*||^2,$$

where $\alpha > 0$ and x^* is a local minimum.

- Let $\epsilon > 0$ be such that $f(x^*) \leq f(x)$ for all feasible x in the *closed* sphere $S = \left\{x \mid ||x - x^*|| \leq \epsilon\right\}$, and let $x^k = \arg\min_{x \in S} F^k(x)$. Have

$$F^k(x^k) = f(x^k) + \frac{k}{2}||h(x^k)||^2 + \frac{\alpha}{2}||x^k - x^*||^2 \leq F^k(x^*) = f(x^*)$$

Hence, $\lim_{k \to \infty} ||h(x^k)|| = 0$, so for every limit point $\overline{x}$ of $\{x^k\}$, $h(\overline{x}) = 0$.

- Furthermore, $f(x^k) + (\alpha/2)||x^k - x^*||^2 \leq f(x^*)$ for all k, so by taking lim,

$$f(\overline{x}) + \frac{\alpha}{2}||\overline{x} - x^*||^2 \leq f(x^*).$$

Combine with $f(x^*) \leq f(\overline{x})$ [since $\overline{x} \in S$ and $h(\overline{x}) = 0$] to obtain $||\overline{x} - x^*|| = 0$ so that $\overline{x} = x^*$. Thus $\{x^k\} \to x^*$.

PENALTY APPROACH - CONTINUED

- Since $x^k \to x^*$, for large k, x^k is interior to S, and is an *unconstrained* local minimum of $F^k(x)$.
- From 1st order necessary condition,

$$0 = \nabla F^k(x^k) = \nabla f(x^k) + k\nabla h(x^k)h(x^k) + \alpha(x^k - x^*). \quad (3)$$

Since $\nabla h(x^*)$ has rank m, $\nabla h(x^k)$ also has rank m for large k, so $\nabla h(x^k)'\nabla h(x^k)$: invertible. Thus, multiplying Eq. (3) w/ $\nabla h(x^k)'$

$$kh(x^k) = -\Big(\nabla h(x^k)'\nabla h(x^k)\Big)^{-1}\nabla h(x^k)'\Big(\nabla f(x^k) + \alpha(x^k - x^*)\Big).$$

Taking limit as $k \to \infty$ and $x^k \to x^*$,

$$\big\{kh(x^k)\big\} \to -\Big(\nabla h(x^*)'\nabla h(x^*)\Big)^{-1}\nabla h(x^*)'\nabla f(x^*) \equiv \lambda^*.$$

Taking limit as $k \to \infty$ in Eq. (3), we obtain

$$\nabla f(x^*) + \nabla h(x^*)\lambda^* = 0.$$

- 2nd order L-multiplier condition: Use 2nd order unconstrained condition for x^k, and algebra.

LAGRANGIAN FUNCTION

- Define the Lagrangian function

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x).$$

Then, if x^* is a local minimum which is regular, the Lagrange multiplier conditions are written

$$\nabla_x L(x^*, \lambda^*) = 0, \qquad \nabla_\lambda L(x^*, \lambda^*) = 0,$$

System of $n + m$ equations with $n + m$ unknowns.

$$y' \nabla^2_{xx} L(x^*, \lambda^*) y \geq 0, \qquad \forall\ y \text{ s.t. } \nabla h(x^*)' y = 0.$$

- Example

minimize $\frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2\right)$

subject to $x_1 + x_2 + x_3 = 3.$

Necessary conditions

$$x_1^* + \lambda^* = 0, \quad x_2^* + \lambda^* = 0,$$

$$x_3^* + \lambda^* = 0, \quad x_1^* + x_2^* + x_3^* = 3.$$

EXAMPLE - PORTFOLIO SELECTION

- Investment of 1 unit of wealth among n assets with random rates of return e_i, and given means $\overline{e}_i$, and covariance matrix $Q = \Big[E\{(e_i - \overline{e}_i)(e_j - \overline{e}_j)\}\Big]$.

- If x_i: amount invested in asset i, we want to

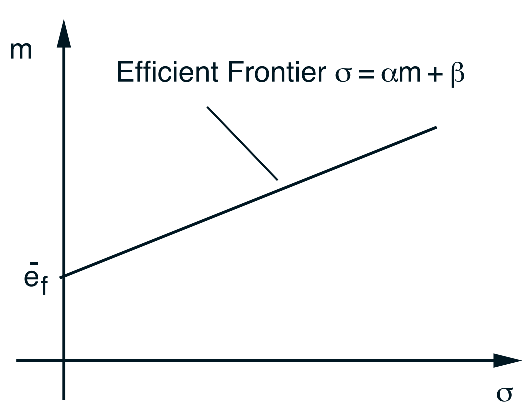
$$\text{minimize} \quad x'Qx \ \left(= \text{Variance of return } \sum_i e_i x_i\right)$$

$$\text{subject to} \quad \textstyle\sum_i x_i = 1, \text{ and a given mean } \displaystyle\sum_i \overline{e}_i x_i = m$$

- Let λ_1 and λ_2 be the L-multipliers. Have $2Qx^* + \lambda_1 u + \lambda_2 \overline{e} = 0$, where $u = (1, \ldots, 1)'$ and $\overline{e} = (\overline{e}_1, \ldots, \overline{e}_n)'$. This yields

$$x^* = mv + w, \quad \text{Variance of return} = \sigma^2 = (\alpha m + \beta)^2 + \gamma,$$

where v and w are vectors, and α, β, and γ are some scalars that depend on Q and $\overline{e}$.



For given m the optimal σ lies on a line (called "efficient frontier").