Problem Set 7 Solutions

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3.1.9

(b) Let the line passing through the points a and b be the set of all x satisfying the equation

$$d'x = c,$$

where d is a vector which can be taken to have unit norm. The projection of a vector x on this line (see Example 2.1.5) is

$$\hat{x} = x - \frac{d'x - c}{\|d\|^2} d = x - (d'x - c)d.$$

Thus the problem of maximizing $\|x - \hat{x}\|^2$ over the unit circle can be expressed as

$$\text{maximize} \quad \|d'x - c\|^2$$

$$\text{subject to} \quad \|x\|^2 = 1.$$

All feasible points of this problem are regular, and there exists a global maximum by Weierstrass' theorem. The first order necessary condition becomes

$$(d'x^* - c)d + \lambda^* x^* = 0$$

which yields

$$(dd' + \lambda^* I)x^* = cd, \qquad \|x^*\|^2 = 1.$$

The solutions to these equations are $(x^*, \lambda^*) = (d, c - 1)$ and $(x^*, \lambda^*) = (-d, -c - 1)$, so the vector x^* is orthogonal to the line connecting the two points. Hence, the line connecting x^* and $\hat{x}$ passes through the center of the circle.

(c) Fix two vertices and the corresponding side of the triangle and optimize over the remaining vertex. By part (b), the line that passes through the optimal vertex and the vertex of the circle should be orthogonal to the fixed side. Hence, the optimal is obtained when the two remaining sides are equal. Repeating this argument, fixing each side of the triangle in turn, we see that all sides should be equal.

(d) Similar line of argument as in (c).

3.1.10

(a) The problem is

$$\text{maximize} \quad f(x) = \|x - a\| + \|x - b\|$$
$$\text{subject to} \|x\|^2 = 1.$$

A global maximum exists by Weierstrass' theorem, and it is not equal to a or b, so the cost function is differentiable at the optimal. By arguing with a Lagrange multiplier as in Fermat's problem, we see that the vectors $x - a$ and $x - b$ must make an equal angle with the constraint gradient, which is a multiple of x.

Consider any triangle inscribed in a given circle, let us fix two of its vertices, say a and b, and let us vary the third vertex x on the circle. By the preceding analysis, the triangle's perimeter $\|x - a\| + \|x - b\| + \|a - b\|$ will be maximized only if the vectors $x - a$ and $x - b$ make an equal angle with x. By elementary geometry this implies that $\|x - a\| = \|x - b\|$ so that the triangle must be isosceles. Thus a maximal perimeter inscribed triangle must have equal sides.

(b) The problem is

$$\text{maximize} \quad f(x) = (x - a)'(x - b)$$
$$\text{subject to} \|x\|^2 = 1.$$

The first order necessary condition is

$$x - a + x - b + 2\lambda x = 0,$$

from which if $a + b \neq 0$, we obtain

$$x = \frac{a + b}{2(1 + \lambda)}$$

where l is such that $\|x\| = 1$ or $2|1 + \lambda| = \|a + b\|$. Thus when $a + b \neq 0$, there are two solutions corresponding to the points where the line defined by the vector $a + b$ intersects the unit circle.

When $a + b = 0$, any feasible x together with $\lambda = -1$ satisfies the first order necessary condition. Thus, to solve the problem in this case, we have to look more closely at the cost function and do some special analysis. Indeed, the cost function is

$$f(x) = (x - a)'(x - b) = \|x\|^2 - (a + b)'x + a'b,$$

so if x is feasible and $a + b = 0$, we have

$$f(x) = 1 + a'b.$$

Thus when $a + b = 0$, *all* feasible points have the same cost and are both global maxima and global minima.

3.2.2

Consider the family of problems

$$\min f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

$$\text{subject to } x_1 + x_2 + x_3 - 3 = u.$$

We have $\nabla h(x) = (\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, so we can apply the Lagrange Multiplier Theorem. The condition $\nabla_x L(x, \lambda) = 0$ is written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

while the condition $\nabla_\lambda L(x, \lambda) = 0$ is written as

$$x_1 + x_2 + x_3 - 3 = u.$$

Solving the system of the above two equations, yields $x(u) = ((u+3)/3, (u+3)/3, (u+3)/3)$ and $\lambda(u) = -(u+3)/3$. Substituting these values into the cost function yields

$$p(u) = f(x(u)) = \frac{1}{2} 3 \left(\frac{u+3}{3} \right)^2 = \frac{(u+3)^2}{6}.$$

So

$$\nabla p(u) = \frac{u+3}{3} = -\lambda(u), \quad \forall\ u.$$

3.4.1

This is just a linear program of the form

$$\max \sum_{i=1}^{n} p_i s_i$$

$$\text{subject to } \sum_{i=1}^{n} s_i = Q, \qquad 0 \le s_i \le d_i, \quad i = 1, \dots, n.$$

Because we are dealing with a convex function over linear constraints, we have a global maximum s^* if and only if

$$s^* = \arg \min_{0 \le s_i \le d_i} \left\{ -\sum_{i=1}^{n} p_i s_i + \lambda^* \left(\sum_{i=1}^{n} s_i - Q \right) \right\}$$

for some λ^*. Now we can use the results from the extension of example 2.1.1 for minimizing over box constraints. Calling

$$L(s) = -\sum_{i=1}^{n} p_i s_i + \lambda^* \left(\sum_{i=1}^{n} s_i - Q \right),$$

we have

$$s_i^* = 0 \Rightarrow \geq 0$$

$$s_i^* = d_i \Rightarrow \leq 0$$

$$0 < s_i^* < d_i \Rightarrow = 0.$$

Taking the gradient of $L(s)$, we see that the result follows, and that the cutoff level y is exactly the Lagrange multiplier λ^*. If $p_i = \lambda^*(= y)$, then we can set s_i to any value in the region $0 \leq s_i^* \leq d_i$ such that we satisfy the resource constraint.

To find s_i^*, we assume without loss of generality that the indices are sorted such that $p_1 \geq p_2 \geq \ldots \geq p_n$, then let k be the largest index such that $\sum_{i=1}^k d_i \leq Q$. If $k < n$, we have $s_i^* = d_i$ for $i = 1, \ldots, k$, $s_{k+1}^* = Q - \sum_{i=1}^k d_i$, and $s_i^* = 0$ for the remaining indices. If $k = n$ then $s_i^* = d_i$ for $i = 1, \ldots, n$. Intuitively, we sell at the best available prices until we have used all of our resources.

3.4.2

The problem is

$$\max \sum_{i=1}^n \frac{p_i x_i}{s_i + x_i} \quad \text{or} \quad \min - \sum_{i=1}^n \frac{p_i x_i}{s_i + x_i}$$

$$\text{subject to } \sum_{i=1}^n x_i = A, \qquad x_i \geq 0, \quad i = 1, \ldots, n.$$

The function $\frac{1}{s_i + x_i}$ is convex over $x_i > -s_i$, and therefore $\frac{x_i}{s_i+x_i} = 1 - \frac{s_i}{s_i+x_i}$ is concave. Hence

$$f(x) = - \sum_{i=1}^n \frac{p_i x_i}{s_i + x_i}$$

is a convex function over $x_i > -s_i$. Then, by Proposition 3.4.2, which holds if the cost function is convex over a region of which the constraint set is in the interior, x^* is a global minimum iff

$$x^* = \arg \min_{x_i \geq 0} \left\{ f(x) - \lambda^* \left(\sum_{i=1}^n x_i - A \right) \right\}$$

for some λ^*. Consider

$$\min_{x_i \geq 0} \left(\sum_{i=1}^n \frac{p_i x_i}{s_i + x_i} - \lambda^* \sum_{i=1}^n x_i \right).$$

Using the results of Example 2.1.1 from Section 2.1, we have

$$x_i^* > 0 \Rightarrow = -\lambda^*,$$

or

$$-\frac{p_i s_i}{(s_i + x_i^*)^2} = -\lambda^*.$$

Thus

$$x_i^* = \sqrt{\frac{p_i s_i}{\lambda^*}} - s_i.$$

Also,

$$\geq -\lambda^* \quad \forall\, i \quad \text{or} \quad \lambda^* - \frac{p_i s_i}{(s_i + x_i^*)^2} \geq 0.$$

So, if $x_i^* = 0$, we have $\lambda^* \geq \frac{p_i}{s_i}$. Thus

$$x_i^* = \begin{cases} \sqrt{\frac{p_i s_i}{\lambda^*}} - s_i, & i = 1, \ldots, m^* \\ 0, & i = m^* + 1, \ldots, n. \end{cases}$$

4.1.1

This is a linear programming problem, with an unique optimal solution

$$x_1^* = 1, x_2^* = 0, x_3^* = 0$$

. To find the central path, we solve the following optimization problem for a given $\epsilon > 0$:

$$\min x_1 + 2x_2 + 3x_3 - \epsilon \sum_{i=1}^{3} \ln x_i$$

$$\text{subject to } x_1 + x_2 + x_3 = 1, x > 0$$

By using Weierstrass' Theorem and strict convexity of the objective function, there is a unique global minimum. That is, for each $\epsilon > 0$, $x(\epsilon)$ is uniquely defined.

We can find the analytical center x_∞ by solving the optimization problem

$$-\epsilon \sum_{i=1}^{3} \ln x_i$$

$$\text{subject to } x_1 + x_2 + x_3 = 1, x > 0$$

It is straight forward to show that $x_\infty = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}'$

The central path ends at x^*, that is $\lim_{\epsilon \to 0} = x^* = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}'$