Problem Set 4 Solutions

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2.1.7

Consider the transformation of variables $y = Q^{1/2}x$ and let $w = Q^{1/2}z$ be the image of the given vector z under this transformation. The problem is equivalent to the problem of projecting w on the closed convex set $Y = \{Q^{1/2}x \mid x \in X\}$, which is the image of X under the transformation (note that Y is closed and convex because X is closed and convex, and $Q^{1/2}$ is invertible).

Now applying the projection theorem, we have that the problem has a unique solution $\hat{w}$ satisfying $(y - \hat{w})'(w - \hat{w}) \leq 0$ for all $y \in Y$. Passing back to the original coordinate system, we have that the unique solution of the original problem is the vector $\hat{z}$ which is such that $\hat{w} = Q^{1/2}\hat{z}$. Furthermore, $\hat{z}$ satisfies the necessary condition $(x - \hat{z})'Q(z - \hat{z}) \leq 0$ for all $x \in X$. A similar argument shows that this condition is also sufficient for optimality.

2.1.13

The objective function is convex, therefore the first order necessary conditions are also sufficient for optimality and every local minima is also global. By the strong convexity of f, we have that x^* is the unique minimum of f in X. Hence $f(x^*) = \min_{x\in X} f(x)$ is equivalent to

$$f(x^*)'(x - x^*) \geq 0, \qquad \forall\, x \in X. \tag{1}$$

By writing down the expression for $f(x^*)$ in (1), we see that (1) is equivalent to

$$\left(\sum_{j=1}^{m}(x^* - a_j)\right)'(x - x^*) \geq 0, \qquad \forall\, x \in X.$$

This is the same as the following

$$\frac{1}{m}\left(\sum_{j=1}^{m}(x^* - a_j)\right)'(x - x^*) \geq 0, \qquad \forall\, x \in X. \tag{2}$$

Finally (2) is equivalent to

$$\left(x^* - \frac{1}{m}\sum_{j=1}^{m} a_j\right)'(x - x^*) \geq 0, \qquad \forall\, x \in X,$$

which means that x^* is the projection of the center of the gravity $\frac{1}{m}\sum_{j=1}^{m} a_j$ on the set X (see Projection Theorem in Ch. 2, Sec. 2.1).