Recitation 2 Solutions

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1.3.1

The Hessian matrix of the function f is

$$Q = \begin{pmatrix} 2 & 1.999 \\ 1.999 & 2 \end{pmatrix}$$

which has largest and smallest eigenvalues $M = 3.999$ and $m = 0.001$ respectively. Hence

$$\frac{f(x_{k+1})}{f(x_k)} \leq \left(\frac{3.998}{4}\right)^2 \approx 0.999$$

Let v_m and v_M be the normalized eigenvectors of Q (see Prop. A.17, Appendix A) corresponding to m and M, respectively, and let

$$x^0 = \frac{s}{m}v_m \pm \frac{s}{M}v_M, \qquad s \in \Re,$$

(cf. Fig. 1.3.2 in Section 1.3). We have

$$x^1 = x^0 - \alpha^0 Q x^0 = (\frac{1}{m} - \alpha^0)sv_m \mp (\frac{1}{M} - \alpha^0)sv_M$$

and

$$f(x^1) = s^2[m(\frac{1}{m} - \alpha^0)^2 + M(\frac{1}{M} - \alpha^0)^2]$$

Using the line minimization stepsize rule, i.e., a stepsize

$$\alpha^* = \arg\min_{\alpha^0}\{s^2[m(\frac{1}{m} - \alpha^0)^2 + M(\frac{1}{M} - \alpha^0)^2]\} = \frac{2}{m+M}$$

, we get the first iteration,

$$x_1 = \left(\frac{M-m}{M+m}\right)\left(\frac{s}{m}v_m \mp \frac{s}{M}v_M\right),$$

which has the same form as x_0 except for the factor $\frac{M-m}{M+m}$. Hence starting the iterations with x_0 we have for all k

$$\frac{f(x_{k+1})}{f(x_k)} = \left(\frac{M-m}{M+m}\right)^2.$$

1.5.2

We have

$$x^{k+1} = y^k - \alpha(y^k - z_2)$$
$$= x^k - \alpha(x^k - z_1) - \alpha(x^k - \alpha(x^k - z_1) - z_2)$$
$$= x^k(1 - 2\alpha + \alpha^2) + \alpha[(1-\alpha)z_1 + z_2] = ax^k + b,$$

where $a = 1 - 2\alpha + \alpha^2$ and $b = \alpha[(1-\alpha)z_1 + z_2]$. Note that since $0 < \alpha < 1$, $0 < a < 1$. Further expanding x^{k+1} yields

$$x^{k+1} = a(ax^{k-1} + b) + b = a^2 x^{k-1} + b(1 + a)$$

$$= a^2(ax^{k-2} + b) + b(1 + a) = a^3 x^{k-2} + b(1 + a + a^2)$$
$$= \ldots + a^{k+1}x^0 + b(1 + a + \ldots + a^k)$$

So

$$\lim_{k\to\infty} x^k = 0 + b\frac{1}{1-a} = \frac{(1-\alpha)z_1 + z_2}{2-\alpha} = x(\alpha).$$

Similarly for y, we have

$$y^{k+1} = x^{k+1} - \alpha(x^{k+1} - z_1) = y^k - \alpha(y^k - z_2) - \alpha(y^k - \alpha(y^k - z_2) - z_1),$$

and so y_{k+1} is related analogously to y_k as x_{k+1} is to x_k. Therefore we have

$$\lim_{k\to\infty} y^k = \frac{(1-\alpha)z_2 + z_1}{2-\alpha} = y(\alpha).$$

From these expressions, it is clear that unless $z_1 = z_2$ or $\alpha = 0$, $x(\alpha) \neq y(\alpha)$, and neither is equal to the optimal least squares solution $x^* = (z_1 + z_2)/2$. However, we do have $x(\alpha) \to x^*$ and $y(\alpha) \to x^*$ as $\alpha \to 0$.

2.1.6

The problem is equivalent to

$$\min -x_1^{a_1} \cdots x_n^{a_n}$$
$$subjectto \textstyle\sum_{i=1}^n x_i = 1, \qquad x_i \geq 0, \quad \forall\ i.$$

From the discussion in Example 2.1.2, the necessary optimality conditions are

$$x_i^* > 0 \qquad \Longrightarrow \qquad \frac{\partial f(x^*)}{\partial x_i} \leq \frac{\partial f(x^*)}{\partial x_j}, \quad \forall\ j$$

or

$$-a_i(x_i^*)^{a_i - 1} \prod_{k\neq i}(x_k^*)^{a_k} \leq -a_j(x_j^*)^{a_j-1} \prod_{k\neq j}(x_k^*)^{a_k}, \quad \forall\ j$$

or

$$a_i x_j^* \geq a_j x_i^*, \quad \forall\ j.$$

It is clear that if x^* is a global minimum, we must have $x_i^* > 0$ for all i. Therefore, the above relation is equivalent to

$$a_i x_j^* = a_j x_i^*, \qquad \forall\ i, j.$$

Summing over all j and using the constraint $\sum_j x_j^* = 1$, we have

$$\sum_j a_i x_j^* = \sum_j a_j x_i^*$$

or

$$a_i \sum_j x_j^* = x_i^* \sum_j a_j$$

or

$$x_i^* = \frac{a_i}{\sum_j a_j}, \qquad \forall\ i.$$

In fact, this is the only point satisfying the necessary conditions. Since the constraint region is compact and the cost function is continuous, a global maximum exists by Weierstrass' theorem, and thus this point is the unique global maximum.