Die Grundlehren
der mathematischen Wissenschaften
in Einzeldarstellungen Band 125

K. Itô H. P. McKean, J

Diffusion Processes and their Sample Paths

Second Printing, Corrected

This book is devoted to the stochastic processes associated with diffusion when now the assumption of spatial homogeneity is dropped . . . Valuable information about Brownian motion is contained in the first two chapters, and the general diffusions (in one or more dimensions of space) then make their appearance. Some information will also be found on a non-linear diffusion of importance in genetics ('the wave of advance of advantageous genes'). The authors become deeply involved with semi-groups of transformations and their sideconditions, and with the precise interpretations of these in terms of the sample paths. This makes the book a formidable one, especially in view of its telegraphic style, but this should not be allowed to obscure the fact that the authors are writing about concrete problems of great importance; moreover, the book contains a very large number of specific examples and may turn out to be the geneticist's Courant and Hilbert of the 1980's. The format and binding are of the highest quality.

D. G. Kendall in "Biometrica"

Markov Processes

By **E. B. Dynkin**

2 volumes, not sold separately
Translation with the authorization and assistance of the author:
J. Fabius, V. Greenberg, A. Maitra, G. Majone

XII, 365 and VIII, 274 pages. 1965
Cloth set DM 96,— ; US $37.00
ISBN 3-540-03301-7
(Grundlehren Bd. 121—122)

The book is in principle self-contained. The pace is nowhere too fast and proofs are spelled out step by step. Nevertheless, prospective readers should be warned that the book is not easy to read for those who have not been exposed to the subject before, since the level of sophistication is high throughout. To the serious student of probability theory however, the book will be invaluable, both as a text and as a reference work.
J. Fabius in "Nieuw Archief voor Wiskunde"

Analytical Treatment of One-Dimensional Markov Processes

By **P. Mandl**

Joint edition with Academia-publishing House of Czechoslovak Academy of Sciences, Prague

XX, 192 pages. 1968
Cloth DM 36,— ; US $13.90
ISBN 3-540-04142-7
(Grundlehren Bd. 151)

The whole set up of the book leans heavily on the work done by Feller, Hille and Ito; the approach to the probability aspects of diffusion processes is purely analytical. A number of interesting new results are given. Undoubtedly, the book is a contribution to theory of diffusion processes.
J. W. Cohen in "Mededelingen van het Wiskundig Genootschap"

Markov Processes

Structure and Asymptotic Behavior

By **M. Rosenblatt**

XIII, 268 pages. 1971
Cloth DM 68,— ; US $26.20
ISBN 3-540-05480-4
(Grundlehren Bd. 184)

The text should be appropriate for an audience with a probabilistic (or statistical) orientation or a more general mathematical (or applied mathematical) outlook. The book can be used as the basis for an interesting course on Markov processes or stationary processes. Many of these topics have not been presented in book form before.

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in Einzeldarstellungen
mit besonderer Berücksichtigung
der Anwendungsgebiete

Band 125

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Diffusion Processes and their Sample Paths

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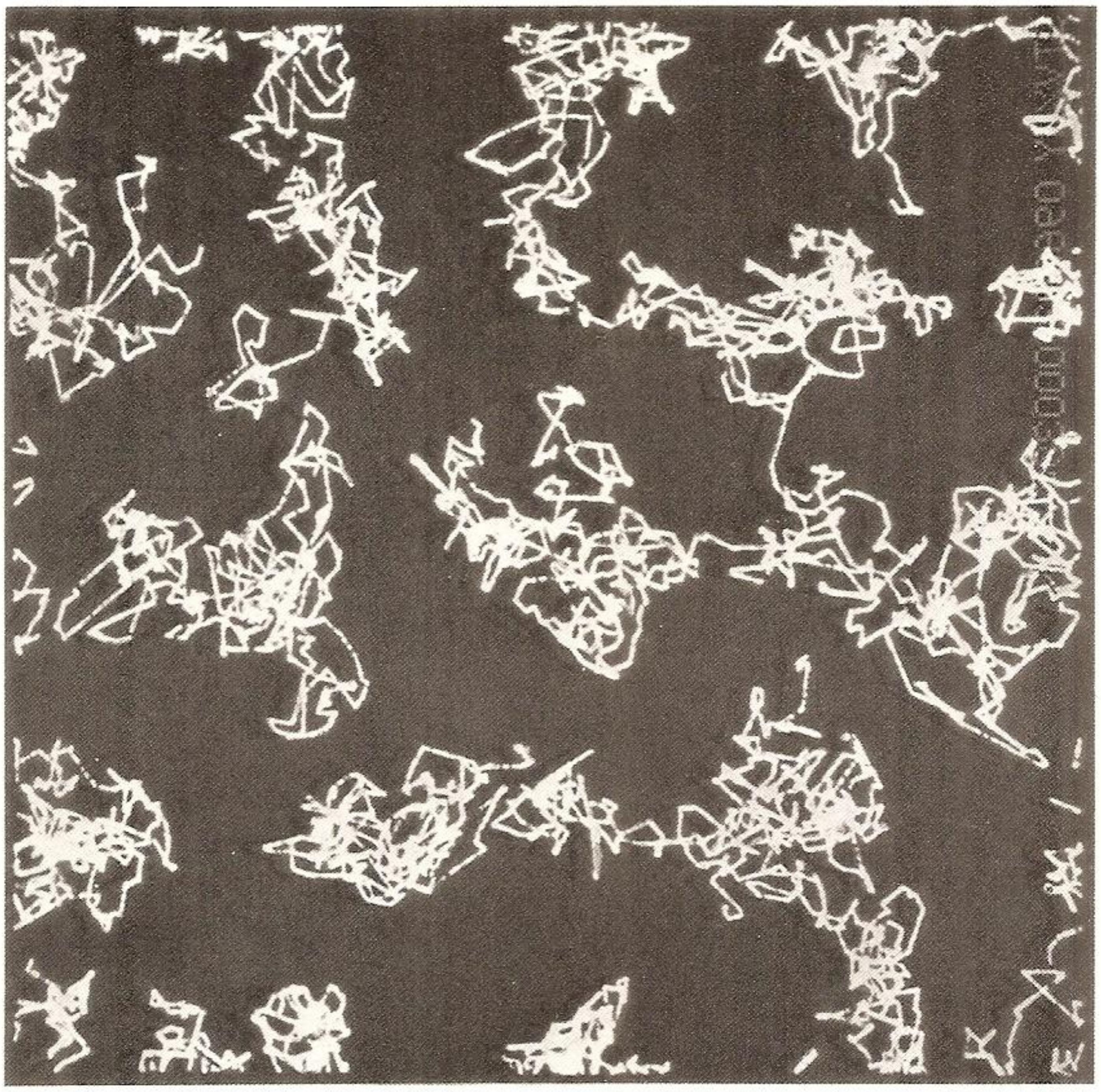

Computer-simulated molecular motions, reminiscent of 2-dimensional BROWNian motion. [From ALDER, B. J., and T. E. WAINWRIGHT: Molecular motions. Scientific American **201**, no. 4, 113—126 (1959)].

Preface

ROBERT BROWN, an English botanist, observed (1828) that pollen grains suspended in water perform a continual swarming motion (see, for example, D'ARCY THOMPSON [*1*: *73*–*77*]).

L. BACHELIER (1900) derived the law govering the position of a single grain performing a 1-dimensional BROWNian motion starting at $a \in R^1$ at time $t = 0$:

1) $$P_a[x(t) \in db] = g(t, a, b)\, db \qquad (t, a, b) \in (0, +\infty) \times R^2,$$

where g is the source (GREEN) function

2) $$g(t, a, b) = \frac{e^{-(b-a)^2/2t}}{\sqrt{2\pi t}}$$

of the problem of heat flow:

3) $$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial a^2} \qquad (t > 0).$$

BACHELIER also pointed out the MARKOVian nature of the BROWNian path expressed in

4) $$\begin{aligned} &P_a[a_1 \leqq x(t_1) < b_1, a_2 \leqq x(t_2) < b_2, \ldots, a_n \leqq x(t_n) < b_n] \\ &= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} g(t_1, a, \xi_1)\, g(t_2 - t_1, \xi_1, \xi_2) \ldots \\ &\quad g(t_n - t_{n-1}, \xi_{n-1}, \xi_n)\, d\xi_1\, d\xi_2 \ldots d\xi_n \qquad 0 < t_1 < t_2 < \cdots < t_n \end{aligned}$$

and used it to establish the law of maximum displacement

5) $$P_0\left[\max_{s \leqq t} x(s) \leqq b\right] = 2 \int_0^b \frac{e^{-a^2/2t}}{\sqrt{2\pi t}}\, da \qquad t > 0, b \geqq 0$$

(see BACHELIER [*1*]).

A. EINSTEIN (1905) also derived 1) from statistical mechanical considerations and applied it to the determination of molecular diameters (see, for example, A. EINSTEIN [*1*]).

BACHELIER was unable to obtain a clear picture of the BROWNian motion and his ideas were unappreciated at the time; nor is this surprising because the precise definition of the BROWNian motion involves a measure on the path space, and it was not until 1909 that E. BOREL published his classical memoir [1] on BERNOULLI trials. But as soon as the ideas of BOREL, LEBESGUE, and DANIELL appeared, it was possible to put the BROWNian motion on a firm mathematical foundation; this was achieved in 1923 by N. WIENER [1].

Consider the space of *continuous* paths $w: t \in [0, +\infty) \to R^1$ with coordinates $x(t) = w(t)$ and let B be the smallest BOREL algebra of subsets B of this path space which includes all the simple events $B = (w: a \leqq x(t) < b)$ $(t \geqq 0, a < b)$. WIENER established the existence of non-negative BOREL measures $P_a(B)$ $(a \in R^1, B \in \mathsf{B})$ for which 4) holds; among other things, this result attaches a precise meaning to BACHELIER's statement that *the Brownian path is continuous.*

P. LÉVY [2] found another construction of the BROWNian motion and, in his 1948 monograph [3], gave a profound description of the fine structure of the individual BROWNian path. LÉVY's results, with several complements due to D. B. RAY [4] and ourselves, will be explained in chapters 1 and 2, with special attention to the standard BROWNian *local time* (the *mesure du voisinage* of P. LÉVY):

6) $$\mathfrak{t}(t, a) = \lim_{b \downarrow a} \frac{\text{measure } (s: a \leqq x(s) < b,\ s \leqq t)}{2(b-a)}.$$

Given a STURM-LIOUVILLE operator $\mathfrak{G} = (c_2/2) D^2 + c_1 D$ $(c_2 > 0)$ on the line, the source (GREEN) function $p = p(t, a, b)$ of the problem

7) $$\frac{\partial u}{\partial t} = \mathfrak{G} u \qquad (t > 0)$$

shares with the GAUSS kernel g of 2) the properties

8a) $$0 \leqq p$$

8b) $$\int_{R^1} p(t, a, b)\, db = 1$$

8c) $$p(t, a, b) = \int_{R^1} p(t-s, a, c)\, p(s, c, b)\, dc \qquad t > s > 0.$$

Soon after the publication of WIENER's monograph [3] in 1930, the associated stochastic motions (diffusions) analogous to the BROWNian motion $(\mathfrak{G} = D^2/2)$ made their debut; the names of W. FELLER and A. N. KOLMOGOROV stand out in this connection. At a later date (1946), K. ITÔ [2] proved that if

9) $$|c_1(b) - c_1(a)| + |\sqrt{c_2(b)} - \sqrt{c_2(a)}| < \text{constant} \times |b - a|,$$

then the motion associated with $\mathfrak{G} = (c_2/2)\, D^2 + c_1\, D$ is identical in law to the *continuous* solution of

$$10) \qquad a(t) = a(0) + \int_0^t c_1(a)\, ds + \int_0^t \sqrt{c_2(a)}\, db$$

where b is a standard BROWNian motion.

W. FELLER took the lead in the next development.

Given a MARKOVian motion with sample paths $w: t \to x(t)$ and probabilities $P_a(B)$ on a linear interval Q, the operators

$$11) \qquad H_t : f \to \int P_a[x(t) \in db]\, f(b)$$

constitute a *semi-group*:

$$12) \qquad H_t = H_{t-s}\, H_s \qquad (t \geqq s),$$

and as E. HILLE [1] and K. YOSIDA [1] proved,

$$13) \qquad H_t = e^{t\mathfrak{G}} \qquad (t > 0)$$

with a suitable interpretation of the exponential, where $\mathfrak{G}$ is the so-called *generator*.

D. RAY [2] proved that if the motion is *strict Markov* (*i.e.*, if it *starts afresh* at certain stochastic (MARKOV) times including the passage times $\mathfrak{m}_a = \min(t : x(t) = a)$, *etc.*), then $\mathfrak{G}$ is *local* if and only if the motion has *continuous* sample paths, substantiating a conjecture of W. FELLER; this combined with FELLER's papers [4, 5, 7, 9] implies that the generator of a strict MARKOVian motion with continuous paths (diffusion) can be expressed as a *differential operator*

$$14) \qquad (\mathfrak{G}\, u)\,(a) = \lim_{b \downarrow a} \frac{u^+(b) - u^+(a)}{m(a,\, b]},$$

where m is a non-negative BOREL measure positive on open intervals and, with a change of scale, $u^+(a) = \lim_{b \downarrow a} (b - a)^{-1} [u(b) - u(a)]$, except at certain singular points where $\mathfrak{G}$ degenerates to a differential operator of degree $\leqq 1$.

E. B. DYNKIN [1] also arrived at the idea of a strict MARKOV process, derived an elegant formula for $\mathfrak{G}$, and used it to make a simple (probabilistic) proof of FELLER's expression for $\mathfrak{G}$; the names of R. BLUMENTHAL [1] and G. HUNT [1] and the monographs of E. B. DYNKIN [6, 8] should also be mentioned in this connection.

Our plan is the following.

BROWNian motion is discussed in chapters 1 and 2 and then, in chapter 3, the general linear *diffusion* is introduced as a strict MARKOVian motion with continuous paths on a linear interval subject to possible annihilation of mass. $\mathfrak{G}$ is computed in great detail in chapter 4 using probabilistic methods similar to those of E. B. DYNKIN [*5*]; in the so-called non-singular case it is a differential operator

15) $$(\mathfrak{G}\, u)\,(a) = \lim_{b \downarrow a} \frac{u^+(b) - u^+(a) - \int\limits_{(a,\, b]} u\, d\, k}{m\,(a,\, b]}$$

with u^+ and m as in 14), where now k is the (non-negative) BOREL measure that governs the annihilation of mass (such generators occur somewhat disguised in W. FELLER [*9*]).

Given $\mathfrak{G}$ as in 14) and a standard BROWNian motion with sample paths $w : t \to x\,(t)$, if $\mathfrak{t}$ is P. LÉVY's *Brownian local time* and if $\mathfrak{f}^{-1}$ is the inverse function of the local time integral

16) $$\mathfrak{f}\,(t) = \int \mathfrak{t}\,(t,\, \xi)\; m\,(d\, \xi)\,,$$

then the motion $x\,(\mathfrak{f}^{-1})$ is identical in law to the diffusion attached to $\mathfrak{G}$, as will be proved in chapter 5, substantiating a suggestion of H. TROTTER; B. VOLKONSKIĬ [*1*] has obtained the same ***time substitution*** in a less explicit form.

Given $\mathfrak{G}$ as in 15), the associated motion can be obtained by ***killing*** the paths $x\,(\mathfrak{f}^{-1})$ described above at a (stochastic) time $\mathfrak{m}_\infty$ with conditional law

17) $$P_\bullet[\mathfrak{m}_\infty > t \mid x(\mathfrak{f}^{-1})] = e^{-\int \mathfrak{t}\,(\mathfrak{f}^{-1}(t),\, \xi)\; k\,(d\, \xi)}$$

as is also proved in chapter 5; in the special case of the *elastic Brownian motion* on $[0, +\infty)$ with generator $\mathfrak{G} = D^2/2$ subject to the condition $\gamma\, u(0) = (1-\gamma)\, u^+(0)$ $(0 < \gamma < 1)$,

$$\mathfrak{f} = \int_0^{+\infty} \mathfrak{t}\,(t,\, \xi)\; 2\, d\, \xi = \text{measure}\,\big(s : x(s) \geqq 0,\; s \leqq t\big)\,,$$

$x\,(\mathfrak{f}^{-1})$ is identical in law to the classical *reflecting Brownian motion* $x^+ = |x|$, and 17) takes the simple form

18) $$P_\bullet[\mathfrak{m}_\infty > t \mid x^+] = e^{-\frac{\gamma}{1-\gamma}\, \mathfrak{t}^+(t,\, 0)} \qquad (\mathfrak{t}^+ = 2\mathfrak{t})\,,$$

substantiating a conjecture of W. FELLER: that the elastic BROWNian motion ought to be the same as the reflecting BROWNian motion killed at the instant a certain increasing functional $e\,(\mathfrak{Z}_+ \cap [0,\, t))$ of the time t and the visiting set $\mathfrak{Z}^+ = (t : x^+(t) = 0)$ hits a certain level.

Details about the fine structure of the sample paths of the general linear diffusion with emphasis on local times, will be found in chapter 6. BROWNian motion in several dimensions is treated in chapter 7, and in chapter 8, the reader will find some glimpses of the general higher-dimensional diffusion.

Acknowledgements. W. FELLER has our best thanks, his ideas run through the whole book, and we shall think it a success if it pleases him.

We have also to thank R. BLUMENTHAL and G. HUNT who placed at our disposal their then unpublished results on MARKOV times, and H. TROTTER with whom we had many helpful conversations.

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Kyôto, Japan, and Cambridge, Mass., November 1964

K. Itô H. P. McKean, Jr.

The present edition is the same as the first except for the correction of numerous errors. Among those who helped us in this task, we would particularly like to thank F. B. KNIGHT.

September 1973 K. I., H. P. McK.

Contents

Numbering: 1.2 means section 2 of chapter 1; 1.2.3) or problem 1.2.3 or diagram 1.2.3 means formula or problem or diagram 3 of 1.2; 3) or problem 3 or diagram 3 means formula or problem or diagram 3 of the *current* section. R. BROWN [*1*: 2] means page 2 of the item R. BROWN [*1*] of the *bibliography*.

Problems accompanied by some indication of their solutions are placed at the end of each section; these often contain additional information needed below and are an essential part of the exposition.

Diagrams do not pretend to photographic faithfulness; for example, the BROWNian path is often pictured as if it had isolated zeros, which is not at all the acual case.

A list of *notations* is placed at the end of the book.

Warning: positive means >0, while non-negative means $\geqq 0$; it is the same for negative (<0) and non-positive ($\leqq 0$). $a \wedge b$ means the smaller of a and b, $a \vee b$ the larger of the two.

Prerequisites

The reader is expected to have about the same mathematical background as is needed to read COURANT-HILBERT [1, 2]. Besides this, he should have mastered most of W. FELLER's book on probability [3] plus the topics listed below (see A. N. KOLMOGOROV [2] for a helpful outline and some of the proofs).

Algebras. Given a space W, a class A of its subsets is said to be an *algebra* if

$$1)\qquad W \in \mathsf{A}$$

$$2)\qquad A - B,\, A \cup B,\, A \cap B \in \mathsf{A} \quad \textit{in case} \quad A, B \in \mathsf{A}.$$

A is said to be a *Borel algebra* if, in addition,

$$3)\qquad \bigcup_{n\geqq 1} B_n,\ \bigcap_{n\geqq 1} B_n \in \mathsf{A} \quad \textit{in case} \quad B_n \in \mathsf{A}\,(n \geqq 1).$$

Borel extension. A class A of subsets of W is contained in a least *Borel algebra* B. B is the so-called *Borel extension* of A.

Probability measures. Consider a non-negative set function $P(C)$ defined on an algebra A. P is said to be a *probability measure* if

$$1)\qquad P(W) = 1$$

and

$$2)\qquad P(A \cup B) = P(A) + P(B) \qquad A, B \in \mathsf{A}, \quad A \cap B = \varnothing.$$

P is said to be a *Borel probability measure* if, in addition,

$$3\,\mathrm{a})\qquad P(B_n) \downarrow 0 \quad (n \uparrow +\infty) \quad B_n \downarrow \varnothing,\, B_n \in \mathsf{A}\,(n \geqq 1)$$

or, what is the same,

$$3\,\mathrm{b})\quad P\Big(\bigcup_{n\geqq 1} B_n\Big) = \sum_{n\geqq 1} P(B_n) \qquad B_n \cap B_m = \varnothing\,(n < m),\quad B_n \in \mathsf{A}(n \geqq 1),\ \bigcup_{n\geqq 1} B_n \in \mathsf{A}.$$

Kolmogorov extension. Given a BOREL probability measure P on an algebra A, there is a unique BOREL probability measure Q on the BOREL extension B of A that coincides with P on A: $Q(B) = \inf \sum_{n\geqq 1} P(A_n)$, where the infemum is taken over all coverings $\bigcup_{n\geqq 1} A_n$ of $B \in \mathsf{B}$ with $A_n \in \mathsf{A}\,(n \geqq 1)$; the triple (W, B, Q) is said to be a *probability space.*

Measurable functions. Given a space W and a BOREL algebra B of its subsets, a function $f: W \to [-\infty, +\infty]$ is said to be *measurable* B (or BOREL) if the ordinate set $f^{-1}[a, b) \in \mathsf{B}$ for each choice of $a < b$.

Integrals. Given a probability space (W, B, Q), the *integral* or *expectation* of a non-negative B measurable function f is defined to be

$$\begin{aligned} E(f) &= \int_W f\, dQ \\ &= \lim_{n\uparrow+\infty} \sum_{l\geqq 1} l\, 2^{-n} Q[f^{-1}[(l-1)\, 2^{-n}, l\, 2^{-n})] \qquad Q[f^{-1}(+\infty)] = 0 \\ &= +\infty \qquad\qquad Q[f^{-1}(+\infty)] > 0. \end{aligned}$$

E, applied to such non-negative functions, satisfies

1) $$E(f) \geqq 0$$

2) $$E(1) = 1$$

3) $$E(f_1 + f_2) = E(f_1) + E(f_2)$$

4a) $$E(f_n) \uparrow E(f) \qquad f_n \uparrow f$$

4b) $$E(f_n) \downarrow E(f) \qquad f_n \downarrow f,\ E(f_1) < +\infty,$$

5) $$E(\underline{\lim}\, f_n) \leqq \underline{\lim}\, E(f_n).$$

$E(f, B) = E(B, f)$ is short for $\int_B f\, dQ$.

Products. Given probability spaces (W_1, B_1, Q_1) and (W_2, B_2, Q_2), the class A of finite sums A of disjoint rectangles $B_1 \times B_2$ ($B_1 \in \mathsf{B}_1$, $B_2 \in \mathsf{B}_2$) is an algebra and

$$Q(B_1 \times B_2) \equiv Q_1(B_1) \times Q_2(B_2)$$

can be extended to a BOREL probability measure on A; the *product* $Q_1 \times Q_2$ is the KOLMOGOROV extension of this Q to the BOREL extension $\mathsf{B} \equiv \mathsf{B}_1 \times \mathsf{B}_2$ of A.

Fubini's theorem. Given a $\mathsf{B}_1 \times \mathsf{B}_2$ measurable function f from $W_1 \times W_2$ to $[0, +\infty)$,

$$\int_{W_1 \times W_2} f\, dQ_1 \times Q_2 = \int_{W_1} dQ_1 \int_{W_2} f\, dQ_2 = \int_{W_2} dQ_2 \int_{W_1} f\, dQ_1.$$

Infinite products. Given probability spaces (W_n, B_n, Q_n) $(n \geqq 1)$,

$$\mathsf{A}_n : A = B \times W_{n+1} \times W_{n+2} \times \textit{etc.}, \quad B \in \mathsf{B}_1 \times \mathsf{B}_2 \times \cdots \times \mathsf{B}_n$$

is a BOREL algebra and

$$Q(A) = Q_1 \times Q_2 \times \cdots \times Q_n(B)$$

is a BOREL probability measure on the algebra $\mathsf{A} = \bigcup_{n\geqq 1} \mathsf{A}_n$; the *infinite product* $\times_{n\geqq 1} Q_n$ is defined as the KOLMOGOROV extension of Q to the BOREL extension $\times_{n\geqq 1} \mathsf{B}_n$ of A.

Independence. Given a probability space (W, B, P), the BOREL algebras $\mathsf{B}_1, \mathsf{B}_2 \subset \mathsf{B}$ are said to be independent if

$$P(B_1 \cap B_2) = P(B_1)\, P(B_2) \qquad B_1 \in \mathsf{B}_1, B_2 \in \mathsf{B}_2;$$

the BOREL algebras B_n $(n \geqq 1)$ are independent, if, for each $n \geqq 1$, B_n is independent of the BOREL extension of $\bigcup_{l \neq n} \mathsf{B}_l$; the sets $B_n \in \mathsf{B}$ $(n \geqq 1)$ are independent if the algebras $\mathsf{B}_n = [\varnothing, B_n, W - B_n, W]$ $(n \geqq 1)$ are such; the B measurable functions f_n $(n \geqq 1)$ are independent if the BOREL extensions F_n of their ordinate sets $f_n^{-1}[a, b)$ $(a < b)$ are such; the B measurable function f is independent of the BOREL algebra $\mathsf{A} \subset \mathsf{B}$ if the BOREL extension of its ordinates sets is independent of A; *etc.* $E(f_1 f_2) = E(f_1)\, E(f_2)$ if f_1 is independent of f_2.

Kolmogorov 01 law. If A_n $(n \geqq 1)$ are independent subalgebras of B, if B_n is the BOREL extension of $\bigcup_{l \geqq n} \mathsf{A}_l$, and if $B \in \bigcap_{n \geqq 1} \mathsf{B}_n$, then $P(B) = 0$ or 1.

Strong law of large numbers. Given independent, non-negative, B measurable functions $f_n (n \geqq 1)$ with the same distribution,

$$P\left[\lim_{n \uparrow +\infty} n^{-1} \sum_{k \leqq n} f_k = E(f_1)\right] = 1.$$

Borel-Cantelli lemmas. Given events $B_n \in \mathsf{B}$ $(n \geqq 1)$, if

$$\sum_{l \geqq 1} P(B_l) < +\infty, \quad \text{then} \quad P\left[\bigcap_{n \geqq 1} \bigcup_{l \geqq n} B_l\right] = 0; \quad \text{if} \quad \sum_{l \geqq 1} P(B_l) = +\infty$$

and if the events are independent, then $P\left[\bigcap_{n \geqq 1} \bigcup_{l \geqq n} B_l\right] = 1$.

Conditional expectations. Given a BOREL subalgebra A of B and a non-negative B measurable function f, *the conditional expectation* $E(f \mid \mathsf{A})$ *of* f *relative to* A is defined to be the class of A measurable functions g such that $E(g, A) = E(f, A)$ $(A \in \mathsf{A})$; two such functions differ on a null set $\in \mathsf{A}$. $E(f \mid \mathsf{A})$ is the RADON-NIKODYM derivative of $Q(A) = E(f, A)$ with respect to the restriction of P to A. $E(f \mid \mathsf{A})$ (applied to non-negative f) satisfies

1) $$E(f \mid \mathsf{A}) \geqq 0$$

2) $$E(1 \mid \mathsf{A}) = 1$$

3) $$E(f_1 + f_2 \mid \mathsf{A}) = E(f_1 \mid \mathsf{A}) + E(f_2 \mid \mathsf{A})$$

4) $$E(f_n \mid \mathsf{A}) \uparrow E(f \mid \mathsf{A}) \qquad f_n \uparrow f$$

5) $$E(E(f \mid \mathsf{A}_2) \mid \mathsf{A}_1) = E(f \mid \mathsf{A}_1) \qquad \mathsf{A}_2 \supset \mathsf{A}_1$$

and

6) $$E(E(f \mid \mathsf{A})) = E(f);$$

in addition,

7) $$E(f \mid \mathsf{A}) = E(f)$$

if f and A are independent, and

8) $$E(e f \mid \mathsf{A}) = e\, E(f \mid \mathsf{A})$$

if e is measurable A. In all this, it is understood that $E(f \mid \mathsf{A}) = (\geqq)\, f_1$ is short for the statement that $f_2 = (\geqq)\, f_1$ for some f_2 from the class $E(f \mid \mathsf{A})$.

Conditional probabilities. Given $B \in \mathsf{B}$, the conditional expectation $E(e \mid \mathsf{A})$ of its indicator function e is the so-called *conditional probability* $P(B \mid \mathsf{A})$ *of* B *relative to* A; the rules for its manipulation are evident from the preceding article.

Gaussian distributions. A class of measurable functions f is said to be *Gaussian* and *centered* if each choice of $x = (f_1, f_2, \ldots, f_d)$ and $(\gamma_1, \ldots, \gamma_d) \in R^d$, the inner product $\gamma \cdot x$ is GAUSS distributed with mean $E(\gamma \cdot x) = 0$, *i.e.*,

1) $$P(a \leqq \gamma \cdot x < b) = \int_a^b \frac{e^{-c^2/2Q}}{\sqrt{2\pi Q}}\, dc,$$

where Q is the quadratic form $Q = Q(\gamma) = E[(\gamma \cdot x)^2] = \sum_{ij} E(f_i f_j)\, \gamma_i \gamma_j$ and the integral is interpreted as the unit mass at $c = 0$ if the determinant $|Q| = 0$. Suppose that the determinant $|Q| \neq 0$; then one can invert the FOURIER transform $E(e^{i\gamma \cdot x})\ (= e^{-Q/2})$ to obtain the probability density of x:

2) $$p(x) = (2\pi)^{-d} \int_{R^d} e^{-i\gamma \cdot x}\, e^{-Q/2}\, d\gamma = \frac{e^{-Q^{-1}(x)/2}}{(2\pi)^{d/2}\sqrt{|Q|}}.$$

where Q^{-1} is the inverse quadratic form. Under the norm $\|f\|_2 = \sqrt{E(f^2)}$, the functions f span out a HILBERT space, and in this geometrical picture, statistical independence and perpendicularity are the same.

Bibliography

AM = Ann. of Math.; BAMS = Bull. Amer. Math. Soc.; IJM = Illinois J. Math.; PNAS = Proc. Nat. Acad. Sci. U.S.A.; TAMS = Trans. Amer. Math. Soc.; TV = Teor. Veroyatnost. i ee Primenen.

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Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete

Eine Auswahl

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167. Chandrasekharan: Arithmetical Functions
168. Palamodov: Linear Differential Operators with Constant Coefficients
170. Lions: Optimal Control Systems Governed by Partial Differential Equations
171. Singer: Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces
172. Bühlmann: Mathematical Methods in Risk Theory
173. F. Maeda/S. Maeda: Theory of Symmetric Lattices
174. Stiefel/Scheifele: Linear and Regular Celestial Mechanics. Perturbed Two-body Motion—Numerical Methods—Canonical Theory
175. Larsen: An Introduction of the Theory of Multipliers
176. Grauert/Remmert: Analytische Stellenalgebren
177. Flügge: Practical Quantum Mechanics I
178. Flügge: Practical Quantum Mechanics II
179. Giraud: Cohomologie non abélienne
180. Landkoff: Foundations of Modern Potential Theory
181. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications I
182. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications II
183. Lions/Magenes: Non-Homogeneous Boundary Value Problems and Applications III
184. Rosenblatt: Markov Processes. Structure and Asymptotic Behavior
185. Rubinowicz: Sommerfeldsche Polynommethode
186. Wilkinson/Reinsch: Handbook for Automatic Computation II. Linear Algebra
187. Siegel/Moser: Lectures on Celestial Mechanics
188. Warner: Harmonic Analysis on Semi-Simple Lie Groups I
189. Warner: Harmonic Analysis on Semi-Simple Lie Groups II
190. Faith: Algebra: Rings, Modules, and Categories I
192. Mal'cev: Algebraic Systems
193. Pólya/Szegö: Problems and Theorems in Analysis. Vol. 1
194. Igusa: Theta Functions
195. Berberian: Baer $*$-Rings
196. Athreya/Ney: Branching Processes
197. Benz: Vorlesungen über Geometrie der Algebren
198. Gaal: Linear Analysis and Representation Theory
200. Dold: Lectures on Algebraic Topology
203. Schoeneberg: Elliptic Modular Functions
206. André: Homologie des algèbres commutatives
207. Donoghue: Monotone Matrix Functions and Analytic Continuation
209. Ringel: Map Color Theorem