Convex Analysis and Optimization

Chapter 7 Solutions

Dimitri P. Bertsekas

with

Angelia Nedić and Asuman E. Ozdaglar

Massachusetts Institute of Technology

Athena Scientific, Belmont, Massachusetts

http://www.athenasc.com

CHAPTER 7: SOLUTION MANUAL

7.1 (Fenchel's Inequality)

Let $f : \Re^n \mapsto (-\infty, \infty]$ be a closed proper convex function and let g be its convex conjugate.

(a) For any $x \in \Re^n$ and $\lambda \in \Re^n$, we have

$$x'\lambda \leq f(x) + g(\lambda).$$

Furthermore, the following are equivalent:

(i) $x'\lambda = f(x) + g(\lambda)$.

(ii) $\lambda \in \partial f(x)$.

(iii) $x \in \partial g(\lambda)$.

(b) The set of minima of f over $\Re^n$ is $\partial g(0)$.

(c) The set of minima of f over $\Re^n$ is nonempty if $0 \in \mathrm{ri}\big(\mathrm{dom}(g)\big)$, and it is nonempty and compact if and only if $0 \in \mathrm{int}\big(\mathrm{dom}(g)\big)$.

Solution: (a) From the definition of g,

$$g(\lambda) = \sup_{x \in \Re^n} \big\{x'\lambda - f(x)\big\},$$

we have the inequality $x'\lambda \leq f(x) + g(\lambda)$. In view of this inequality, the equality $x'\lambda = f(x) + g(\lambda)$ of (i) is equivalent to the inequality

$$x'\lambda - f(x) \geq g(\lambda) = \sup_{z \in \Re^n} \big\{z'\lambda - f(z)\big\},$$

or

$$x'\lambda - f(x) \geq z'\lambda - f(z), \qquad \forall\ z \in \Re^n,$$

or

$$f(z) \geq f(x) + \lambda'(z - x), \qquad \forall\ z \in \Re^n,$$

which is equivalent to (ii). Since f is closed, f is equal to the conjugate of g, so by using the equivalence of (i) and (ii) with the roles of f and g reversed, we obtain the equivalence of (i) and (iii).

(b) A vector x^* minimizes f if and only if $0 \in \partial f(x^*)$, which by part (a), is true if and only if $x^* \in \partial g(0)$.

(c) The result follows by combining part (b) and Prop. 4.4.2.

7.2

Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper convex function, and let g be its conjugate. Show that the lineality space of g is equal to the orthogonal complement of the subspace parallel to $\text{aff}\big(\text{dom}(f)\big)$.

Solution: Let $f : \Re^n \mapsto (-\infty, \infty]$ be a proper convex function, and let g be its conjugate. Show that the lineality space of g is equal to the orthogonal complement of the subspace parallel to $\text{aff}\big(\text{dom}(f)\big)$.

7.3

Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be proper convex functions, and let $f = f_1 + \cdots + f_m$. Show that if $\cap_{i=1}^m \text{ri}\big(\text{dom}(f_i)\big)$ is nonempty, then we have

$$g(\lambda) = \inf_{\substack{\lambda_1 + \cdots + \lambda_m = \lambda \\ \lambda_i \in \Re^n,\, i=1,\ldots,m}} \big\{ g_1(\lambda_1) + \cdots + g_m(\lambda_m) \big\}, \qquad \forall\, \lambda \in \Re^n,$$

where $g, g_1, \ldots, g_m$ are the conjugates of $f, f_1, \ldots, f_m$, respectively.

Solution: Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i = 1, \ldots, m$, be proper convex functions, and let $f = f_1 + \cdots + f_m$. Show that if $\cap_{i=1}^m \text{ri}\big(\text{dom}(f_i)\big)$ is nonempty, then we have

$$g(\lambda) = \inf_{\substack{\lambda_1 + \cdots + \lambda_m = \lambda \\ \lambda_i \in \Re^n,\, i=1,\ldots,m}} \big\{ g_1(\lambda_1) + \cdots + g_m(\lambda_m) \big\}, \qquad \forall\, \lambda \in \Re^n,$$

where $g, g_1, \ldots, g_m$ are the conjugates of $f, f_1, \ldots, f_m$, respectively.

7.4 (Finiteness of the Optimal Dual Value)

Consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \qquad g_j(x) \le 0, \quad j = 1, \ldots, r, \end{aligned}$$

where X is a convex set, and f and g_j are convex over X. Assume that the problem has at least one feasible solution. Show that the following are equivalent.

(i) The dual optimal value $q^* = \sup_{\mu \ge 0} q(\mu)$ is finite.

(ii) The primal function p is proper.

(iii) The set

$$M = \big\{ (u, w) \in \Re^{r+1} \mid \text{there is an } x \in X \text{ such that } g(x) \le u,\ f(x) \le w \big\}$$

does not contain a vertical line.

Solution: Consider the function $\tilde{q}$ given by

$$\tilde{q}(\mu) = \begin{cases} q(\mu) & \text{if } \mu \geq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

and note that $-\tilde{q}$ is closed and convex, and that by the calculation of Example 7.1.5, we have

$$\tilde{q}(\mu) = \inf_{u \in \Re^r} \big\{ p(u) + \mu' u \big\}, \qquad \forall\ \mu \in \Re^r. \tag{1}$$

Since $\tilde{q}(\mu) \leq p(0)$ for all $\mu \in \Re^r$, given the feasibility of the problem [i.e., $p(0) < \infty$], we see that q^* is finite if and only if $-\tilde{q}$ is proper. From Eq. (1), $-\tilde{q}$ is the conjugate of $p(-u)$, and by the Conjugacy Theorem [Prop. 7.1.1(b)], $-\tilde{q}$ is proper if and only if p is proper. Hence, (i) is equivalent to (ii).

We note that the epigraph of p is the closure of M. Hence, given the feasibility of the problem, (ii) is equivalent to the closure of M not containing a vertical line. Since M is convex, its closure does not contain a line if and only if M does not contain a line (since the closure and the relative interior of M have the same recession cone). Hence (ii) is equivalent to (iii).

7.5 (General Perturbations and Min Common/Max Crossing Duality)

Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a proper function, and let $G : \Re^{n+m} \mapsto (-\infty, \infty]$ be its conjugate. Let also p be the function defined by

$$p(u) = \inf_{x \in \Re^n} F(x, u), \qquad u \in \Re^m.$$

Consider the min common/max crossing framework for the set $M = \text{epi}(p)$, and the cost function of the max crossing problem, $q(\lambda) = \inf_{(u,w) \in M} \{w + \lambda' u\}$.

(a) Show that q and the conjugate h of p satisfy

$$h(\lambda) = G(0, \lambda), \qquad q(\lambda) = -G(0, -\lambda), \qquad \forall\ \lambda \in \Re^m.$$

Show also that these relations generalize Example 7.1.5.

(b) Consider the alternative min common/max crossing framework where

$$M = \big\{ (u, w) \mid \text{there is an } x \text{ such that } F(x, u) \leq w \big\}.$$

Show that the optimal values of the corresponding min common and max crossing problems are the same as those corresponding to $M = \text{epi}(p)$.

(c) Show that with $F(x, u) = f_1(x) - f_2(Qx + u)$, the min common/max crossing framework corresponds to the Fenchel duality framework. What are the forms of F that correspond to the minimax and constrained optimization frameworks of Sections 2.6.1 and 6.1?

Solution: (a) We have

$$\begin{aligned} h(\lambda) &= \sup_u \big\{ \lambda' u - p(u) \big\} \\ &= \sup_u \big\{ \lambda' u - \inf_x F(x,u) \big\} \\ &= \sup_{x,u} \big\{ \lambda' u - F(x,u) \big\} \\ &= G(0,\lambda). \end{aligned}$$

Also

$$\begin{aligned} q(\lambda) &= \inf_{(u,w)\in M} \{ w + \lambda' u \} \\ &= \inf_{x,u} \big\{ F(x,u)'_\lambda u \big\} \\ &= -\sup_{x,u} \big\{ -\lambda' u - F(x,u) \big\} \\ &= -G(0,-\lambda). \end{aligned}$$

Consider the constrained minimization propblem of Example 7.1.5:

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \qquad g(x) \le 0, \end{aligned}$$

and define

$$F(x,u) = \begin{cases} f(x) & \text{if } x \in X \text{ and } g(x) \le u, \\ \infty & \text{otherwise.} \end{cases}$$

Then p is the primal function of the constrained minimization problem. Consider now $q(\lambda)$, the cost function of the max crossing problem corresponding to M. For $\lambda \ge 0$, $q(\lambda)$ is equal to the dual function value of the constrained optimization problem, and otherwise $q(\lambda)$ is equal to $-\infty$. Thus, the relations $h(\lambda) = G(0,\lambda)$ and $q(\lambda) = -G(0,-\lambda)$ proved earlier, show the relation proved in Example 7.1.5, i.e., that $q(\lambda) = -h(-\lambda)$.

(b) Let

$$M = \big\{ (u,w) \mid \text{there is an } x \text{ such that } F(x,u) \le w \big\}.$$

Then the corresponding min common value is

$$\inf_{\{(x,w) \mid F(x,0) \le w\}} w = \inf_x F(x,0) = p(0).$$

Since $p(0)$ is the min common value corresponding to epi(p), the min common values corresponding to the two choises for M are equal. Similarly, we show that the cost functions of the max crossing problem corresponding to the two choises for M are equal.

(c) If $F(x,u) = f_1(x) - f_2(Qx + u)$, we have

$$p(u) = \inf_x \big\{ f_1(x) - f_2(Qx + u) \big\},$$

so $p(0)$, the min common value, is equal to the primal optimal value in the Fenchel duality framework. By part (a), the max crossing value is

$$q^* = \sup_\lambda \big\{ -h(-\lambda) \big\},$$

where h is the conjugate of p. By using the change of variables $z = Qx + u$ in the following calculation, we have

$$\begin{aligned} -h(-\lambda) &= -\sup_u \Big\{ -\lambda' u - \inf_x \big\{ f_1(x) - f_2(Qx+u) \big\} \Big\} \\ &= -\sup_{z,x} \big\{ -\lambda'(z - Qx) - f_1(x) + f_2(z) \big\} \\ &= g_2(\lambda) - g_1(Q\lambda), \end{aligned}$$

where g_1 and g_2 are the conjugate convex and conjugate concave functions of f_1 and f_2, respectively:

$$g_1(\lambda) = \sup_x \big\{ x'\lambda - f_1(x) \big\}, \qquad g_2(\lambda) = \inf_z \big\{ z'\lambda - f_2(z) \big\}.$$

Thus, no duality gap in the min common/max crossing framework [i.e., $p(0) = q^* = \sup_\lambda \big\{ -h(-\lambda) \big\}$] is equivalent to no duality gap in the Fenchel duality framework.

The minimax framework of Section 2.6.1 (using the notation of that section) is obtained for

$$F(x,u) = \sup_{z \in Z} \big\{ \phi(x,z) - u'z \big\}.$$

The constrained optimization framework of Section 6.1 (using the notation of that section) is obtained for the function

$$F(x,u) = \begin{cases} f(x) & \text{if } x \in X,\ h(x) = u_1,\ g(x) \le u_2, \\ \infty & \text{otherwise,} \end{cases}$$

where $u = (u_1, u_2)$.

7.6

Use Minimax Theorem III (Prop. 3.5.3) to derive the following version of the Primal Fenchel Duality Theorem: Let the functions f_1 and $-f_2$ be proper and convex. Then we have

$$\inf_{x \in \Re^n} \big\{ f_1(x) - f_2(x) \big\} = \sup_{\lambda \in \Re^m} \big\{ g_2(\lambda) - g_1(\lambda) \big\},$$

and the supremum in the right-hand side above is attained, if

$$\mathrm{ri}\big(\mathrm{dom}(f_1)\big) \cap \mathrm{ri}\big(\mathrm{dom}(-f_2)\big) \neq \emptyset.$$

Hint: In view of the results of Exercise 1.35, it is sufficient to show the above equality when f_1 and $-f_2$ are replaced by their closures.

Solution: By Exercise 1.35,

$$\mathrm{cl}\, f_1 + \mathrm{cl}\,(-f_2) = \mathrm{cl}\,(f_1 - f_2).$$

Furthermore,

$$\inf_{x\in\Re^n} \mathrm{cl}\,(f_1 - f_2)(x) = inf_{x\in\Re^n}\big(f_1(x) - f_2(x)\big).$$

Thus, we may replace f_1 and $-f_2$ with their closures, and the result follows by applying Minimax Theorem III.

7.7 (Monotropic Programming Duality)

Consider the problem

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^n f_i(x_i) \\ &\text{subject to} \quad x \in S, \qquad x_i \in X_i, \quad i = 1,\ldots,n, \end{aligned}$$

where $f_i : \Re \mapsto \Re$ are given functions, X_i are intervals of real numbers, and S is a subspace of $\Re^n$. Assume that the problem is feasible and that its optimal value is finite.

(a) Show that a dual problem is

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^n g_i(\lambda_i) \\ &\text{subject to} \quad \lambda \in S^\perp, \end{aligned}$$

where the functions $g_i : \Re \mapsto (-\infty,\infty]$ are the conjugate convex functions

$$g_i(\lambda_i) = \sup_{x_i\in X_i} \big\{\lambda_i x_i - f_i(x_i)\big\}, \qquad i = 1,\ldots,n.$$

(b) Show that the dual problem has an optimal solution and there is no duality gap under one of the following two conditions:

(1) Each function f_i is convex over X_i and S contains a point in the relative interior of $X_1 \times \cdots \times X_n$.

(2) The intervals X_i are closed and the functions f_i are convex over the entire real line.

Solution: We apply Fenchel duality with

$$f_1(x) = \begin{cases} \sum_{i=1}^n f_i(x_i) & \text{if } x \in X_1 \times \cdots \times X_n, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{if } x \in S, \\ -\infty & \text{otherwise.} \end{cases}$$

The corresponding conjugate concave and convex functions g_2 and g_1 are

$$\inf_{x \in S} \lambda' x = \begin{cases} 0 & \text{if } \lambda \in S^\perp, \\ -\infty & \text{if } \lambda \notin S^\perp, \end{cases}$$

where $S^\perp$ is the orthogonal subspace of S, and

$$\sup_{x_i \in X_i} \left\{ \sum_{i=1}^n \big(x_i \lambda_i - f_i(x_i) \big) \right\} = \sum_{i=1}^n g_i(\lambda_i),$$

where for each i,

$$g_i(\lambda_i) = \sup_{x_i \in X_i} \big\{ x_i \lambda_i - f_i(x_i) \big\}.$$

By the Primal Fenchel Duality Theorem (Prop. 7.2.1), the dual problem has an optimal solution and there is no duality gap if the functions f_i are convex over X_i and one of the following two conditions holds:

(1) The subspace S contains a point in the relative interior of $X_1 \times \cdots \times X_n$.

(2) The intervals X_i are closed (so that the Cartesian product $X_1 \times \cdots \times X_n$ is a polyhedral set) and the functions f_i are convex over the entire real line.

These conditions correspond to the two conditions for no duality gap given following Prop. 7.2.1.

7.8 (Network Optimization and Kirchhoff's Laws)

Consider a linear resistive electric network with node set $\mathcal{N}$ and arc set $\mathcal{A}$. Let v_i be the voltage of node i and let x_{ij} be the current of arc (i, j). Kirchhoff's current law says that for each node i, the total outgoing current is equal to the total incoming current

$$\sum_{\{j \mid (i,j) \in \mathcal{A}\}} x_{ij} = \sum_{\{j \mid (j,i) \in \mathcal{A}\}} x_{ji}.$$

Ohm's law says that the current x_{ij} and the voltage drop $v_i - v_j$ along each arc (i, j) are related by

$$v_i - v_j = R_{ij} x_{ij} - t_{ij},$$

where $R_{ij} \geq 0$ is a resistance parameter and t_{ij} is another parameter that is nonzero when there is a voltage source along the arc (i, j) (t_{ij} is positive if the voltage source pushes current in the direction from i to j). Consider the problem

$$\begin{aligned} &\text{minimize} && \sum_{(i,j) \in \mathcal{A}} \left(\tfrac{1}{2} R_{ij} x_{ij}^2 - t_{ij} x_{ij} \right) \\ &\text{subject to} && \sum_{\{j \mid (i,j) \in \mathcal{A}\}} x_{ij} = \sum_{\{j \mid (j,i) \in \mathcal{A}\}} x_{ji}, \quad \forall\, i \in \mathcal{N}. \end{aligned} \tag{7.0}$$

Show that a set of variables $\{x_{ij} \mid (i,j) \in \mathcal{A}\}$ and $\{v_i \mid i \in \mathcal{N}\}$ are an optimal solution-Lagrange multiplier pair for this problem if and only if they satisfy Kirchhoff's current law and Ohm's law.

Solution: This problem is a monotropic programming problem, as considered in Exercise 7.7. For each $(i,j) \in \mathcal{A}$, the function $f_{ij}(x_{ij}) = \frac{1}{2}R_{ij}x_{ij}^2 - t_{ij}x_{ij}$ is continuously differentiable and convex over $\Re$. The dual problem is

$$\begin{aligned} &\text{maximize } q(v) \\ &\text{subject to no constraints on } p, \end{aligned}$$

with the dual function q given by

$$q(v) = \sum_{(i,j)\in\mathcal{A}} q_{ij}(v_i - v_j),$$

where

$$q_{ij}(v_i - v_j) = \min_{x_{ij}\in\Re} \left\{ \frac{1}{2}R_{ij}x_{ij}^2 - (v_i - v_j + t_{ij})x_{ij} \right\}.$$

Since the primal cost functions f_{ij} are real-valued and convex over the entire real line, there is no duality gap. The necessary and sufficient conditions for a set of variables $\{x_{ij} \mid (i,j) \in \mathcal{A}\}$ and $\{v_i \mid i \in \mathcal{N}\}$ to be an optimal solution-Lagrange multiplier pair are:

(1) The set of variables $\{x_{ij} \mid (i,j) \in \mathcal{A}\}$ must be primal feasible, i.e., Kirchhoff's current law must be satisfied.

(2)
$$x_{ij} \in \arg\min_{y_{ij}\in\Re} \left\{ \frac{1}{2}R_{ij}y_{ij}^2 - (v_i - v_j + t_{ij})y_{ij} \right\}, \quad \forall\ (i,j) \in \mathcal{A},$$

which is equivalent to Ohm's law:

$$R_{ij}x_{ij} - (v_i - v_j + t_{ij}) = 0, \quad \forall\ (i,j) \in \mathcal{A}.$$

Hence a set of variables $\{x_{ij} \mid (i,j) \in \mathcal{A}\}$ and $\{v_i \mid i \in \mathcal{N}\}$ are an optimal solution-Lagrange multiplier pair if and only if they satisfy Kirchhoff's current law and Ohm's law.

7.9 (Symmetry of Duality)

Consider the primal function

$$p(u) = \inf_{x\in X,\, g(x)\le u} f(x)$$

of the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \qquad g_j(x) \le 0, \quad j = 1, \ldots, r. \end{aligned} \tag{7.1}$$

Consider also the problem

$$
\begin{aligned}
&\text{minimize} \quad p(u) \\
&\text{subject to} \quad u \in P, \qquad u \le 0,
\end{aligned}
\tag{7.2}
$$

where P is the effective domain of p,

$$P = \{u \mid \text{there exists } x \in X \text{ with } g(x) \le u\}.$$

Assume that $-\infty < p(0) < \infty$.

(a) Show that problems (7.1) and (7.2) have equal optimal values, and the same sets of geometric multipliers.

(b) Consider the dual functions of problems (7.1) and (7.2) and show that they are equal on the positive orthant, i.e., for all $\mu \ge 0$,

$$q(\mu) = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j g_j(x) \right\} = \inf_{u \in P} \{p(u) + \mu' u\}.$$

(c) Assume that p is a closed and convex function. Show that u^* is an optimal solution of problem (7.2) if and only if $-u^*$ is a geometric multiplier for the dual problem

$$
\begin{aligned}
&\text{maximize} \quad q(\mu) \\
&\text{subject to} \quad \mu \ge 0,
\end{aligned}
$$

in the sense that

$$q^* = \sup_{\mu \ge 0} \{q(\mu) - \mu' u^*\}.$$

Solution: (a) We have $f^* = p(0)$. Since $p(u)$ is monotonically nonincreasing, its minimal value over $u \in P$ and $u \le 0$ is attained for $u = 0$. Hence, $f^* = p^*$, where $p^* = \inf_{u \in P,\, u \le 0} p(u)$. For $\mu \ge 0$, we have

$$
\begin{aligned}
\inf_{x \in X} \{f(x) + \mu' g(x)\} &= \inf_{u \in P} \inf_{x \in X,\, g(x) \le u} \{f(x) + \mu' g(x)\} \\
&= \inf_{u \in P} \{p(u) + \mu' u\}.
\end{aligned}
$$

Since $f^* = p^*$, we see that $f^* = \inf_{x \in X}\{f(x) + \mu' g(x)\}$ if and only if $p^* = \inf_{u \in P}\{p(u) + \mu' u\}$. In other words, the two problems have the same geometric multipliers.

(b) This part was proved by the preceding argument.

(c) From Example 7.1.5, we have that $-q(-\mu)$ is the conjugate convex function of p. Let us view the dual problem as the minimization problem

$$
\begin{aligned}
&\text{minimize} \quad -q(-\mu) \\
&\text{subject to} \quad \mu \le 0.
\end{aligned}
\tag{1}
$$

Its dual problem is obtained by forming the conjugate convex function of its primal function, which is p, based on the analysis of Example 7.1.5, and the closedness and convexity of p. Hence the dual of the dual problem (1) is

$$\begin{aligned} &\text{maximize} \quad -p(u) \\ &\text{subject to} \quad u \le 0 \end{aligned}$$

and the optimal solutions to this problem are the geometric multipliers to problem (1).

7.10 (Second-Order Cone Programming)

Consider the problem

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad \|A_j x + b_j\| \le e_j' x + d_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $x \in \Re^n$, and c, A_j, b_j, e_j, and d_j are given, and have appropriate dimension. Assume that the problem is feasible. Consider the equivalent problem

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \quad \|u_j\| \le t_j, \quad u_j = A_j x + b_j, \quad t_j = e_j' x + d_j, \quad j = 1, \ldots, r, \end{aligned} \tag{7.3}$$

where u_j and t_j are auxiliary optimization variables.

(a) Show that problem (7.3) has cone constraints of the type described in Section 7.2.2.

(b) Use the conic duality theory of Section 7.2.2 to show that a dual problem is given by

$$\begin{aligned} &\text{minimize} \quad \sum_{j=1}^{r} (b_j' z_j + d_j w_j) \\ &\text{subject to} \quad \sum_{j=1}^{r} (A_j' z_j + e_j w_j) = c, \qquad \|z_j\| \le w_j, \quad j = 1, \ldots, r. \end{aligned} \tag{7.4}$$

Furthermore, show that there is no duality gap if either there exists a feasible solution of problem (7.3) or a feasible solution of problem (7.4) satisfying strictly all the corresponding inequality constraints.

Solution: (a) Define

$$X = \Big\{(x, u, t) \mid x \in \Re^n,\ u_j = A_j x + b_j,\ t_j = e_j' x + d_j,\ j = 1, \ldots, r\Big\},$$

$$C = \Big\{(x, u, t) \mid x \in \Re^n,\ \|u_j\| \le t_j,\ j = 1, \ldots, r\Big\}.$$

It can be seen that X is convex and C is a cone. Therefore the modified problem can be written as

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X \cap C, \end{aligned}$$

and is a cone programming problem of the type described in Section 7.2.2.

(b) Let $(\lambda, z, w) \in \hat{C}$, where $\hat{C}$ is the dual cone ($\hat{C} = -C^*$, where C^* is the polar cone). Then we have

$$\lambda' x + \sum_{j=1}^{r} z_j' u_j + \sum_{j=1}^{r} w_j t_j \geq 0, \qquad \forall\ (x, u, t) \in C.$$

Since x is unconstrained, we must have $\lambda = 0$ for otherwise the above inequality will be violated. Furthermore, it can be seen that

$$\hat{C} = \Big\{(0, z, w) \mid \|z_j\| \leq w_j,\ j = 1, \ldots, r\Big\}.$$

By the conic duality theory of Section 7.2.2, the dual problem is given by

$$\begin{aligned} &\text{minimize} \ \sum_{j=1}^{r} (z_j' b_j + w_j d_j) \\ &\text{subject to} \ \sum_{j=1}^{r} (A_j' z_j + w_j e_j) = c, \quad \|z_j\| \leq w_j,\ j = 1, \ldots, r. \end{aligned}$$

If there exists a feasible solution of the modified primal problem satisfying strictly all the inequality constraints, then the relative interior condition $\text{ri}(X) \cap \text{ri}(C) \neq \emptyset$ is satisfied, and there is no duality gap. Similarly, if there exists a feasible solution of the dual problem satisfying strictly all the inequality constraints, there is no duality gap.

7.11 (Quadratically Constrained Quadratic Problems [LVB98])

Consider the quadratically constrained quadratic problem

$$\begin{aligned} &\text{minimize} \quad x' P_0 x + 2q_0' x + r_0 \\ &\text{subject to} \quad x' P_i x + 2q_i' x + r_i \leq 0, \quad i = 1, \ldots, p, \end{aligned}$$

where $P_0, P_1, \ldots, P_p$ are symmetric positive definite matrices. Show that the problem can be converted to one of the type described in Exercise 7.10, and derive the corresponding dual problem. *Hint*: Consider the equivalent problem

$$\begin{aligned} &\text{minimize} \quad \|P_0^{1/2} x + P_0^{-1/2} q_0\| \\ &\text{subject to} \quad \|P_i^{1/2} x + P_i^{-1/2} q_i\| \leq (r_i - q_i' P_i^{-1} q_i)^{1/2}, \quad i = 1, \ldots, p. \end{aligned}$$

Solution: Since each P_i is symmetric and positive definite, we have

$$\begin{aligned} x'P_ix + 2q_i'x + r_i &= \left(P_i^{1/2}x\right)' P_i^{1/2}x + 2\left(P_i^{-1/2}q_i\right)' P_i^{1/2}x + r_i \\ &= \|P_i^{1/2}x + P_i^{-1/2}q_i\|^2 + r_i - q_i'P_i^{-1}q_i, \end{aligned}$$

for $i = 0, 1, \ldots, p$. This allows us to write the original problem as

$$\begin{aligned} &\text{minimize} \ \ \|P_0^{1/2}x + P_0^{-1/2}q_0\|^2 + r_0 - q_0'P_0^{-1}q_0 \\ &\text{subject to} \ \ \|P_i^{1/2}x + P_i^{-1/2}q_i\|^2 + r_i - q_i'P_i^{-1}q_i \le 0, \ i = 1, \ldots, p. \end{aligned}$$

By introducing a new variable x_{n+1}, this problem can be formulated in $\Re^{n+1}$ as

$$\begin{aligned} &\text{minimize} \ \ x_{n+1} \\ &\text{subject to} \ \ \|P_0^{1/2}x + P_0^{-1/2}q_0\| \le x_{n+1} \\ &\|P_i^{1/2}x + P_i^{-1/2}q_i\| \le \left(q_i'P_i^{-1}q_i - r_i\right)^{1/2}, \quad i = 1, \ldots, p. \end{aligned}$$

The optimal values of this problem and the original problem are equal up to a constant and a square root. The above problem is of the type described in Exercise 7.10. To see this, define $A_i = \left(P_i^{1/2} \mid 0\right)$, $b_i = P_i^{-1/2}q_i$, $e_i = 0$, $d_i = \left(q_i'P_i^{-1}q_i - r_i\right)^{1/2}$ for $i = 1, \ldots, p$, $A_0 = \left(P_0^{1/2} \mid 0\right)$, $b_0 = P_0^{-1/2}q_0$, $e_0 = (0, \ldots, 0, 1)$, $d_0 = 0$, and $c = (0, \ldots, 0, 1)$. Its dual is given by

$$\begin{aligned} &\text{maximize} \ -\sum_{i=1}^{p}\left(q_i'P_i^{-1/2}z_i + \left(q_i'P_i^{-1}q_i - r_i\right)^{1/2} w_i\right) - q_0'P_0^{-1/2}z_0 \\ &\text{subject to} \ \sum_{i=0}^{p} P_i^{1/2}z_i = 0, \quad \|z_0\| \le 1, \quad \|z_i\| \le w_i, \quad i = 1, \ldots, p. \end{aligned}$$

7.12 (Minimizing the Sum or the Maximum of Norms [LVB98])

Consider the problems

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{p} \|F_ix + g_i\| \\ &\text{subject to} \ \ x \in \Re^n, \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} &\text{minimize} \quad \max_{i=1,\ldots,p} \|F_ix + g_i\| \\ &\text{subject to} \ \ x \in \Re^n, \end{aligned}$$

where F_i and g_i are given matrices and vectors, respectively. Convert these problems to second-order cone programming problems (cf. Exercise 7.10) and derive the corresponding dual problems.

Solution: Consider the problem

$$\begin{aligned} &\text{minimize } \sum_{i=1}^{p} ||F_i x + g_i|| \\ &\text{subject to } x \in \Re^n. \end{aligned}$$

By introducing variables $t_1, \ldots, t_p$, this problem can be expressed as a second-order cone programming problem (see Exercise 7.10):

$$\begin{aligned} &\text{minimize } \sum_{i=1}^{p} t_i \\ &\text{subject to } ||F_i x + g_i|| \le t_i, \quad i = 1, \ldots, p. \end{aligned}$$

Define

$$X = \{(x, u, t) \mid x \in \Re^n,\ u_i = F_i x + g_i,\ t_i \in \Re,\ i = 1, \ldots, p\},$$

$$C = \{(x, u, t) \mid x \in \Re^n,\ ||u_i|| \le t_i, \quad i = 1, \ldots, p\}.$$

Then, similar to Exercise 7.10, we have

$$-C^* = \{(0, z, w) \mid ||z_i|| \le w_i, \quad i = 1, \ldots, p\},$$

and

$$\begin{aligned} g(0, z, w) &= \sup_{(x,u,t) \in X} \left\{ \sum_{i=1}^{p} z_i' u_i + \sum_{i=1}^{p} w_i t_i - \sum_{i=1}^{p} t_i \right\} \\ &= \sup_{x \in \Re^n, t \in \Re^p} \left\{ \sum_{i=1}^{p} z_i'(F_i x + g_i) + \sum_{i=1}^{p} (w_i - 1) t_i \right\} \\ &= \sup_{x \in \Re^n} \left\{ \left(\sum_{i=1}^{p} F_i' z_i \right)' x \right\} + \sup_{t \in \Re^p} \left\{ \sum_{i=1}^{p} (w_i - 1) t_i \right\} + \sum_{i=1}^{p} g_i' z_i \\ &= \begin{cases} \sum_{i=1}^{p} g_i' z_i & \text{if } \sum_{i=1}^{p} F_i' z_i = 0,\ w_i = 1,\ i = 1, \ldots, p \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the dual problem is given by

$$\begin{aligned} &\text{maximize } -\sum_{i=1}^{p} g_i' z_i \\ &\text{subject to } \sum_{i=1}^{p} F_i' z_i = 0, \quad ||z_i|| \le 1,\ i = 1, \ldots, p. \end{aligned}$$

Now, consider the problem

$$\begin{aligned}&\text{minimize } \max_{1\le i\le p} \|F_i x + g_i\|\\&\text{subject to } x \in \Re^n.\end{aligned}$$

By introducing a new variable x_{n+1}, we obtain

$$\begin{aligned}&\text{minimize } x_{n+1}\\&\text{subject to } \|F_i x + g_i\| \le x_{n+1},\ i = 1,\ldots,p,\end{aligned}$$

or equivalently

$$\begin{aligned}&\text{minimize } e'_{n+1}x\\&\text{subject to } \|A_i x + g_i\| \le e'_{n+1}x,\ i = 1,\ldots,p,\end{aligned}$$

where $x \in \Re^{n+1}$, $A_i = (F_i, 0)$, and $e_{n+1} = (0,\ldots,0,1)' \in \Re^{n+1}$. Evidently, this is a second-order cone programming problem. From Exercise 7.10 we have that its dual problem is given by

$$\begin{aligned}&\text{maximize } -\sum_{i=1}^{p} g_i' z_i\\&\text{subject to } \sum_{i=1}^{p}\left(\begin{pmatrix} F_i' \\ 0\end{pmatrix} z_i + e_{n+1}w_i\right) = e_{n+1}, \quad \|z_i\| \le w_i, \quad i = 1,\ldots,p,\end{aligned}$$

or equivalently

$$\begin{aligned}&\text{maximize } -\sum_{i=1}^{p} g_i' z_i\\&\text{subject to } \sum_{i=1}^{p} F_i' z_i = 0, \quad \sum_{i=1}^{p} w_i = 1, \quad \|z_i\| \le w_i, \quad i = 1,\ldots,p.\end{aligned}$$

7.13 (Complex l_1 and l_∞ Approximation [LVB98])

Consider the complex l_1 approximation problem

$$\begin{aligned}&\text{minimize} \quad \|Ax - b\|_1\\&\text{subject to} \quad x \in \mathcal{C}^n,\end{aligned}$$

where $\mathcal{C}^n$ is the set of n-dimensional vectors whose components are complex numbers. Show that it is a special case of problem (7.5) and derive the corresponding dual problem. Repeat for the complex l_∞ approximation problem

$$\begin{aligned}&\text{minimize} \quad \|Ax - b\|_\infty\\&\text{subject to} \quad x \in \mathcal{C}^n.\end{aligned}$$

Solution:
For $v \in \mathcal{C}^p$ we have

$$\|v\|_1 = \sum_{i=1}^{p} |v_i| = \sum_{i=1}^{p} \left\| \begin{pmatrix} \mathcal{R}e(v_i) \\ \mathcal{I}m(v_i) \end{pmatrix} \right\|,$$

where $\mathcal{R}e(v_i)$ and $\mathcal{I}m(v_i)$ denote the real and the imaginary parts of v_i, respectively. Then the complex l_1 approximation problem is equivalent to

$$\begin{aligned} &\text{minimize} \ \sum_{i=1}^{p} \left\| \begin{pmatrix} \mathcal{R}e(a_i'x - b_i) \\ \mathcal{I}m(a_i'x - b_i) \end{pmatrix} \right\| \\ &\text{subject to } x \in \mathcal{C}^n, \end{aligned} \tag{1}$$

where a_i' is the i-th row of A (A is a $p \times n$ matrix). Note that

$$\begin{pmatrix} \mathcal{R}e(a_i'x - b_i) \\ \mathcal{I}m(a_i'x - b_i) \end{pmatrix} = \begin{pmatrix} \mathcal{R}e(a_i') & -\mathcal{I}m(a_i') \\ \mathcal{I}m(a_i') & \mathcal{R}e(a_i') \end{pmatrix} \begin{pmatrix} \mathcal{R}e(x) \\ \mathcal{I}m(x) \end{pmatrix} - \begin{pmatrix} \mathcal{R}e(b_i) \\ \mathcal{I}m(b_i) \end{pmatrix}.$$

By introducing new variables $y = (\mathcal{R}e(x'), \mathcal{I}m(x'))'$, problem (1) can be rewritten as

$$\begin{aligned} &\text{minimize} \ \sum_{i=1}^{p} \|F_i y + g_i\| \\ &\text{subject to } y \in \Re^{2n}, \end{aligned}$$

where

$$F_i = \begin{pmatrix} \mathcal{R}e(a_i') & -\mathcal{I}m(a_i') \\ \mathcal{I}m(a_i') & \mathcal{R}e(a_i') \end{pmatrix}, \qquad g_i = -\begin{pmatrix} \mathcal{R}e(b_i) \\ \mathcal{I}m(b_i) \end{pmatrix}. \tag{2}$$

According to Exercise 7.12, the dual problem is given by

$$\begin{aligned} &\text{maximize} \ \sum_{i=1}^{p} \big(\mathcal{R}e(b_i), \mathcal{I}m(b_i) \big) z_i \\ &\text{subject to} \ \sum_{i=1}^{p} \begin{pmatrix} \mathcal{R}e(a_i') & \mathcal{I}m(a_i') \\ -\mathcal{I}m(a_i') & \mathcal{R}e(a_i') \end{pmatrix} z_i = 0, \qquad \|z_i\| \le 1, \quad i = 1, \dots, p, \end{aligned}$$

where $z_i \in \Re^{2n}$ for all i.

For $v \in \mathcal{C}^p$ we have

$$\|v\|_\infty = \max_{1 \le i \le p} |v_i| = \max_{1 \le i \le p} \left\| \begin{pmatrix} \mathcal{R}e(v_i) \\ \mathcal{I}m(v_i) \end{pmatrix} \right\|.$$

Therefore the complex l_∞ approximation problem is equivalent to

$$\begin{aligned} &\text{minimize} \ \max_{1 \le i \le p} \left\| \begin{pmatrix} \mathcal{R}e(a_i'x - b_i) \\ \mathcal{I}m(a_i'x - b_i) \end{pmatrix} \right\| \\ &\text{subject to } x \in \mathcal{C}^n. \end{aligned}$$

By introducing new variables $y = (\mathcal{R}e(x'), \mathcal{I}m(x'))'$, this problem can be rewritten as

$$\begin{aligned} &\text{minimize } \max_{1\le i\le p} \|F_i y + g_i\| \\ &\text{subject to } y \in \Re^{2n}, \end{aligned}$$

where F_i and g_i are given by Eq. (2). From Exercise 7.12, it follows that the dual problem is

$$\begin{aligned} &\text{maximize } \sum_{i=1}^{p} \big(\mathcal{R}e(b_i), \mathcal{I}m(b_i)\big) z_i \\ &\text{subject to } \sum_{i=1}^{p} \begin{pmatrix} \mathcal{R}e(a_i') & -\mathcal{I}m(a_i') \\ \mathcal{I}m(a_i') & \mathcal{R}e(a_i') \end{pmatrix} z_i = 0, \quad \sum_{i=1}^{p} w_i = 1, \quad \|z_i\| \le w_i, \\ &\qquad\qquad i = 1, \ldots, p, \end{aligned}$$

where $z_i \in \Re^2$ for all i.

7.14

Consider the case where in Prop. 7.3.1 the function P has the form

$$P(z) = \sum_{j=1}^{r} P_j(z_j),$$

where $P_j : \Re \mapsto \Re$ are convex real-valued functions satisfying

$$P_j(z_j) = 0, \quad \forall\, z_j \le 0, \qquad P_j(z_j) > 0, \quad \forall\, z_j > 0.$$

Show that the conditions 7.22 and 7.23 of Prop. 7.3.1 are equivalent to

$$\mu_j^* \le \lim_{z_j \downarrow 0} \frac{P_j(z_j)}{z_j}, \qquad \forall\, j = 1, \ldots, r,$$

and

$$\mu_j^* < \lim_{z_j \downarrow 0} \frac{P_j(z_j)}{z_j}, \qquad \forall\, j = 1, \ldots, r,$$

respectively.

Solution: The condition $u'\mu^* \le P(u)$ for all $u \in \Re^r$ can be written as

$$\sum_{j=1}^{r} u_j \mu_j^* \le \sum_{j=1}^{r} P_j(u_j), \qquad \forall\, u = (u_1, \ldots, u_r),$$

and is equivalent to

$$u_j \mu_j^* \le P_j(u_j), \qquad \forall\, u_j \in \Re,\ \forall\, j = 1, \ldots, r.$$

In view of the requirement that P_j is convex with $P_j(u_j) = 0$ for $u_j \le 0$, and $P_j(u_j) > 0$ for all $u_j > 0$, it follows that the condition $u_j \mu_j^* \le P_j(u_j)$ for all $u_j \in \Re$, is equivalent to $\mu_j^* \le \lim_{z_j \downarrow 0}\big(P_j(z_j)/z_j\big)$. Similarly, the condition $u_j \mu_j^* < P_j(u_j)$ for all $u_j \in \Re$, is equivalent to $\mu_j^* < \lim_{z_j \downarrow 0}\big(P_j(z_j)/z_j\big)$.

7.15 [Ber99b]

For a vector $y \in \Re^n$, let $d(y)$ be the optimal value of the projection problem

$$\begin{aligned} &\text{minimize} \quad \|y - x\| \\ &\text{subject to} \;\; x \in X, \qquad g_j(x) \le 0, \qquad j = 1, \ldots, r, \end{aligned}$$

where X is a convex subset of $\Re^n$, and the functions $g_j : \Re^n \mapsto \Re$ are convex over X. Assuming that the problem is feasible, show that there exists a constant c such that

$$d(y) \le c \big\| \big(g(y)\big)^+ \big\|, \qquad \forall\, y \in X,$$

if and only if the projection problem has a geometric multiplier $\mu^*(y)$ such that the set $\big\{\mu^*(y) \mid y \in X\big\}$ is bounded.

Solution: Following [Ber99b], we address the problem by embedding it in a broader class of problems. Let Y be a subset of $\Re^n$, let y be a parameter vector taking values in Y, and consider the parametric program

$$\begin{aligned} &\text{minimize} \;\; f(x, y) \\ &\text{subject to} \;\; x \in X, \qquad g_j(x, y) \le 0, \quad j = 1, \ldots, r, \end{aligned} \tag{1}$$

where X is a convex subset of $\Re^n$, and for each $y \in Y$, $f(\cdot, y)$ and $g_j(\cdot, y)$ are real-valued functions that are convex over X. We assume that for each $y \in Y$, this program has a finite optimal value, denoted by $f^*(y)$. Let $c > 0$ denote a penalty parameter and assume that the penalized problem

$$\begin{aligned} &\text{minimize} \;\; f(x, y) + c\big\|g^+(x, y)\big\| \\ &\text{subject to} \;\; x \in X \end{aligned} \tag{2}$$

has a finite optimal value, thereby coming under the framework of Section 7.3. By Prop. 7.3.1, we have

$$f^*(y) = \inf_{x \in X} \big\{ f(x, y) + c\big\|g^+(x, y)\big\| \big\}, \qquad \forall\, y \in Y, \tag{3}$$

if and only if

$$u'\mu^*(y) \le c\|u^+\|, \qquad \forall\, u \in \Re^r,\ \forall\, y \in Y,$$

for some geometric multiplier $\mu^*(y)$.

It is seen that Eq. (3) is equivalent to the bound

$$f^*(y) \le f(x, y) + c\big\|g^+(x, y)\big\|, \qquad \forall\, x \in X,\ \forall\, y \in Y, \tag{4}$$

so this bound holds if and only if there exists a uniform bounding constant $c > 0$ such that

$$u'\mu^*(y) \le c\|u^+\|, \qquad \forall\, u \in \Re^r,\ \forall\, y \in Y. \tag{5}$$

Thus the bound (4), holds if and only if for every $y \in Y$, it is possible to select a geometric multiplier $\mu^*(y)$ of the parametric problem (1) such that the set $\big\{\mu^*(y) \mid y \in Y\big\}$ is bounded.

Let us now specialize the preceding discussion to the parametric program

$$\begin{aligned} &\text{minimize} \quad f(x,y) = \|y - x\| \\ &\text{subject to} \quad x \in X, \qquad g_j(x) \le 0, \quad j = 1, \ldots, r, \end{aligned} \tag{6}$$

where $\|\cdot\|$ is the Euclidean norm, X is a convex subset of $\Re^n$, and g_j are convex over X. This is the projection problem of the exercise. Let us take $Y = X$. If c satisfies Eq. (5), the bound (4) becomes

$$d(y) \le \|y - x\| + c\big\|\big(g(x)\big)^+\big\|, \qquad \forall\, x \in X,\ \forall\, y \in X,$$

and (by taking $x = y$) implies the bound

$$d(y) \le c\big\|\big(g(y)\big)^+\big\|, \qquad \forall\, y \in X. \tag{7}$$

This bound holds if a geometric multiplier $\mu^*(y)$ of the projection problem (6) can be found such that Eq. (5) holds. We will now show the reverse assertion.

Indeed, assume that for some c, Eq. (7) holds, and to arrive at a contradiction, assume that there exist $x \in X$ and $y \in Y$ such that

$$d(y) > \|y - x\| + c\big\|\big(g(x)\big)^+\big\|.$$

Then, using Eq. (7), we obtain

$$d(y) > \|y - x\| + d(x).$$

From this relation and the triangle inequality, it follows that

$$\begin{aligned} \inf_{z \in X,\, g(z) \le 0} \|y - z\| &> \|y - x\| + \inf_{z \in X,\, g(z) \le 0} \|x - z\| \\ &= \inf_{z \in X,\, g(z) \le 0} \big\{\|y - x\| + \|x - z\|\big\} \\ &\ge \inf_{z \in X,\, g(z) \le 0} \|y - z\|, \end{aligned}$$

which is a contradiction. Thus Eq. (7) implies that we have

$$d(y) \le \|y - x\| + c\big\|\big(g(x)\big)^+\big\|, \qquad \forall\, x \in X,\ \forall\, y \in X.$$

Using Prop. 7.3.1, this implies that there exists a geometric multiplier $\mu^*(y)$ such that

$$u'\mu^*(y) \le c\|u^+\|, \qquad \forall\, u \in \Re^r,\ \forall\, y \in X.$$

This in turn implies the boundedness of the set $\big\{\mu^*(y) \mid y \in X\big\}$.