Algebraic Topology

The Fundamental Group

Homotopy

Given two maps $f_1, f_2 : X \to Y$, a *homotopy* from f_1 to f_2 is a map $F : [0,1] \times X \to Y$ such that $F(0,x) = f_1(x)$ and $F(1,x) = f_2(x)$. If there is a homotopy from f_1 to f_2 then we say that f_1 and f_2 are *homotopic* and we write $f_1 \sim f_2$. Homotopy is an equivalence relation, although to prove transitivity we need the following lemma:

The 'Gluing Lemma'

If a space X is a union of two closed subsets A and B then for any two continuous maps $f : A \to Y$ and $g : B \to Y$ which agree on the intersection $A \cap B$ we get a continuous map $f \cup g : X \to Y$.

This is easy to prove. The set of *homotopy classes* of maps $X \to Y$ is written $[X, Y]$.

Relative homotopy

Given two maps $f_1, f_2 : X \to Y$ which agree on a subset $A \subset X$, we say that they are homotopic *relative* to A if there is a map $F : [0,1] \times X \to Y$ such that $F(0,x) = f_1(x)$, $F(1,x) = f_2(x)$ and $F(x,t) = f_0(x)$ for all $x \in A$.

The fundamental group

The *fundamental group* of X based at x, written $\pi_1(X,x)$, is the group of homotopy classes relative to $\{0,1\}$ of loops in X based at x, with the natural multiplication given by composition of loops. It is easy to show that the multiplication is well-defined and satisfies the group axioms.

Note that for loops p, q, we write pq to mean the loop which goes round p first and then q.

Induced homomorphisms of fundamental groups

If we have a map $f : X \to Y$ then we get an induced homomorphism $f_* : \pi_1(X,x) \to \pi_1(Y, f(x))$ between the fundamental groups of X and Y, by mapping a loop p in X to the loop fp in Y. It is easy to check that this is a group homomorphism.

Also, if we have the identity map $1_X : X \to X$ then $(1_X)_*$ is the identity homomorphism, and for any two maps $f : X \to Y$ and $g : Y \to Z$ then we have $(g \circ f)_* = g_* \circ f_*$. Thus if f is a homeomorphism $X \to Y$ then the induced homomorphism f_* is an isomorphism, and hence the fundamental group is a topological invariant.

If two maps $f, g : X \to Y$ are homotopic relative to $\{x\}$ then the induced homomorphisms f_* and g_* are the same, since the homotopy F from f to g determines a homotopy $F(\alpha(s), t)$ from $f(\alpha)$ to $g(\alpha)$ for any loop α in X based at x. This is a special case of a result below.

Change of base point

If $x, y \in X$, then $\pi_1(X,x)$ and $\pi_1(X,y)$ are unrelated if x and y are in different path-components, but isomorphic if they are in the same path component. For any path α from x to y we get a group isomorphism $\alpha_* : \pi_1(X, x) \to \pi_1(X, y)$. Note that in general different paths α may lead to different isomorphisms.

Contractible spaces

A non-empty space X is said to be *simply connected* if it is path-connected and has trivial fundamental group. X is said to be *contractible* if the identity map $X \to X$ is homotopic to a constant map. For example, any convex subset of $\mathbb{R}^n$ is contractible.

A contractible space is simply connected. Note that the converse is not true — S^2 is simply connected but not contractible.

To prove this, we need to show that any loop in X is homotopic, relative to $\{0, 1\}$, to the constant loop. Observe that the homotopy F from the identity on X to the constant map at $x \in X$ gives us a homotopy $G(s, t) = F(\alpha(s), t)$ from any loop α based at x to the constant loop at x — but this homotopy is not necessarily relative to $\{0, 1\}$.

To get around this difficulty we use the fact that the square $[0, 1] \times [0, 1]$ is convex, so there is a straight line homotopy H between any two paths from a to b in $[0, 1] \times [0, 1]$, and so any two such paths are homotopic relative to $\{0, 1\}$. Using this, we see that the path along the left edge of the square is homotopic, relative to $\{0, 1\}$, to the path along the bottom edge, up the right edge and back along the top edge of the square. Composing the homotopies H and G gives us the result.

Homotopy equivalence

Two spaces X and Y are *homotopy equivalent* if there is a map $f : X \to Y$ and a map $g : Y \to X$ such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. Such an f is called a *homotopy equivalence*. We also say that X and Y have the same *homotopy type*, and we write $X \simeq Y$. The relation $X \simeq Y$ is an equivalence relation.

Any homeomorphism is clearly a homotopy equivalence. It is also very easy to prove that a space is contractible if and only if it is homotopy equivalent to a point.

Two homotopy equivalent spaces have the same number of path-components. Furthermore, two homotopy equivalent spaces have isomorphic fundamental groups.

To prove this, we first show that if f and g are homotopic maps from X to Y and $x \in X$, then $g_* = \alpha_* \circ f_*$ as maps from $\pi_1(X, x)$ to $\pi_1(Y, g(x))$, where α is the path in Y from $f(x)$ to $g(x)$ given by the homotopy F between f and g. This follows by using the homotopy F to show that for any loop α in X based at x, the loops $g_*(\alpha)$ and $(\alpha_* \circ f_*)(\alpha)$ in Y are homotopic.

It then follows that the map f_* is an isomorphism of fundamental groups if and only if g_* is. Therefore, if f is a homotopy equivalence with homotopy inverse h, then $f \circ h \simeq 1_X$ and so $(f \circ h)_* = f_* \circ h_*$ is an isomorphism. Thus f_* is injective. Working the other way we see that f_* is surjective. Hence f_* gives an isomorphism between the fundamental groups of X and Y.

The fundamental group of the n-sphere

If X is a space which can be written as the union of two simply connected open subsets A and B in such a way that $A \cap B$ is path-connected, then X is simply connected. To prove this we show that every loop in X starting at a point p in $A \cap B$ is homotopic to a (finite) product of loops, each of which is either in A or in B.

Using this result, we see that the fundamental group of S^n is trivial for $n \geq 2$.

Covering Spaces

Fundamental group of the circle

To determine the fundamental group of the circle we first define the map $\pi : \mathbb{R} \to S^1$ by $\pi(x) = e^{2\pi ix}$. π is a covering map as defined below. We then apply the path-lifting and homotopy-lifting lemmas related to this covering map.

We define a homomorphism $\sigma : \pi_1(S^1, 1) \to \mathbb{Z}$ as follows. Let α be a loop in S^1 based at 1. This lifts uniquely to a path in $\mathbb{R}$ starting at 0, whose endpoint must be an integer, and we define $\sigma(\alpha)$ to be this value.

To show that σ gives a well-defined map on $\pi_1(S^1, 1)$ we use the homotopy-lifting property. For if $\beta \sim \alpha$ is another loop in S^1 based at 1 then the homotopy from α to β gives us a homotopy from the lift of α to the lift of β, and we then see that $\sigma(\alpha) = \sigma(\beta)$.

It is easy to see that σ is a homomorphism. Furthermore, σ is injective, for if $\sigma(\alpha) = 0$ then there is a straight line homotopy, relative to $\{0, 1\}$, from the lift of α to the constant loop at 0 in $\mathbb{R}$, and this projects to show that α is homotopic to the constant loop in $\pi_1(S^1, 1)$. Finally, σ is surjective, for the loop $s \mapsto e^{2\pi ins}$ in S^1 lifts to a path in $\mathbb{R}$ from 0 to n.

Therefore we have shown that the fundamental group of the circle is isomorphic to $\mathbb{Z}$.

The fundamental theorem of algebra

Let f be a polynomial of degree $n \geq 1$, and suppose that f has no roots in $\mathbb{C}$. Then f is a continuous map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$. Since $\mathbb{C} \setminus \{0\}$ and S^1 are homotopy equivalent, they have isomorphic fundamental groups. In particular, loops in $\mathbb{C} \setminus \{0\}$ have a well-defined winding number, which is invariant under homotopy of loops (even if you change the base point).

Now let C_r be the circle of radius r around 0 in $\mathbb{C}$, and consider the image of C_r under f as r varies. These circles are obviously all homotopic, and a homotopy from C_{r_1} to C_{r_2} gives a homotopy from $f \circ C_{r_1}$ to $f \circ C_{r_2}$. Now the circle C_0 clearly projects to a single point, with winding number 0, but for sufficiently large R the image $f \circ C_R$ is homotopic to $(a_n x^n) \circ C_R$, which has winding number n. This contradicts the homotopy invariance of the winding number.

Covering maps and covering spaces

A *covering map* $\pi : X \to Y$ is a continuous map such that every $y \in Y$ has an open neighbourhood U such that $\pi^{-1}(U)$ is a disjoint union of open subsets U_α such that the restriction of π to each set U_α is a homeomorphism from U_α to U. We then say that X is a *covering space* of Y.

The path-lifting property

For any covering space $X \to Y$ and any point $x \in X$, any path in Y starting at $\pi(x)$ lifts to a unique path in X starting at x.

The path-lifting property is a special case of the homotopy-lifting property below, where the space A is a point.

The homotopy-lifting property

For any covering space $X \to Y$, any other space A and any continuous map $f' : A \to X$, any homotopy from $f = \pi f' : A \to Y$ to some other map $g : A \to Y$ lifts uniquely to a homotopy from f' to some other map $g' : A \to X$.

Computing fundamental groups using covering spaces

For any covering space $\pi : X \to Y$ and any point $x \in X$, the induced group homomorphism $\pi_* : \pi_1(X, x) \to \pi_1(Y, \pi(x))$ is injective. This is proved using the homotopy-lifting property.

If we have a covering space $\pi : X \to Y$, then $\pi_1(Y, y)$ acts naturally on the fibre $\pi^{-1}(y)$. Furthermore, if X is path-connected then this action is transitive. In this case, the stabilizer of a point $x \in \pi^{-1}(y)$ is precisely the fundamental group $\pi_1(X, x)$ of X, and we have a bijection from the set of cosets of $\pi_1(X, x)$ in $\pi_1(Y, y)$ to the fibre $\pi^{-1}(y)$ given by

$$\pi_1(X, x)\,\sigma \longmapsto x\sigma.$$

Therefore we can compute the order of the fundamental group of Y if we know the order of the fundamental group of X. To determine the structure we need the following theorem:

Let X be a simply connected space. Let G be a group of homeomorphisms which acts freely on X in the sense that every point $x \in X$ has an open neighbourhood U such that $U \cap g(U) = \emptyset$ for all $g \neq 1 \in G$. Let $Y = X/G$ be the space of orbits. Then the map $X \to Y$ is a covering map and the fundamental group of Y at any base point is isomorphic to G.

Fundamental group of the torus

The n-torus $(S^1)^n$ has fundamental group $\mathbb{Z}^n$. We can prove this in two ways. Firstly, we can use the fact that

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

Alternatively, we observe that $(S^1)^n$ is the quotient of $\mathbb{R}^n$ by the free action of $\mathbb{Z}^n$ acting by translation, and that $\mathbb{R}^n$ is simply connected, so the result follows by the theorem above.

Fundamental group of real projective space

The real projective space $\mathbb{RP}^n$ can be thought of as the quotient of S^n by the free action of the group $\mathbb{Z}/2$, mapping each point in S^n to its antipode. Therefore, if $n = 0$ then $\mathbb{RP}^0$ consists of just one point, and hence is simply connected. If $n = 1$ then $\mathbb{RP}^1$ is isomorphic to S^1 and so has fundamental group $\mathbb{Z}$. If $n \geq 2$ then S^n is simply connected, and so the fundamental group of $\mathbb{RP}^n$ is $\mathbb{Z}/2$.

The universal covering

A space is *locally contractible* if every point has an open neighbourhood which is contractible. Also, we say that two covering spaces X_1 and X_2 of the same space Y are *isomorphic* if there is a homeomorphism from X_1 to X_2 such that the composition $X_1 \to X_2 \to Y$ is the map $X_1 \to Y$. Then we have the following theorem:

Every connected, locally contractible space Y has a unique simply connected covering space X, called the *universal covering* of Y. Moreover, the fundamental group of Y acts freely on X, with $Y = X/\pi_1(Y, y)$.

Therefore, the fundamental group of any reasonable space Y may be computed by finding a simply connected covering space of Y and applying this theorem. Finally, the following theorem classifies the connected covering spaces of a space Y.

Let Y be a connected, locally contractible space. Then there is a bijection between isomorphism classes of connected covering spaces of Y and conjugacy classes of subgroups H of the group $G = \pi_1(Y, y)$. The correspondence is defined by viewing Y as X/G, where X is the universal cover of Y, and, for each subgroup H of G, forming the new covering space X/H of Y.

Simplicial Complexes

Simplices and simplicial complexes

We say that $n+1$ points in $\mathbb{R}^N$ are in *general position* if the smallest affine-linear subspace containing them has dimension n. An *n-simplex* Δ in $\mathbb{R}^N$ is the convex hull of any $n+1$ points $x_0, \ldots, x_n \in \mathbb{R}^N$ in general position, that is

$$\Delta = \{a_0 x_0 + \cdots + a_n x_n \mid a_i \in \mathbb{R}, a_i \geq 0, \sum a_i = 1\}.$$

We call $x_0, \ldots, x_n$ the *vertices* of the simplex. A *face* of the simplex is the convex hull of a non-empty subset of the vertices.

A *simplicial complex* in $\mathbb{R}^N$ is a finite collection of simplices in $\mathbb{R}^N$, such that whenever a simplex belongs to the collection then so do all of its faces, and whenever any two simplices in the collection have a non-empty intersection, their intersection is a face of both simplices.

If X is a simplicial complex we write $|X|$ to mean the *geometric realisation* of X, that is the topological space which is the union of all the simplices in X. $|X|$ is always a compact, metrizable space.

Triangulations

For a topological space A, a *triangulation* of A is a simplicial complex X together with a homeomorphism from A to $|X|$. We say that A is *triangulable* if it has some triangulation.

Barycentric subdivision

We define the *barycentre* of the simplex $[x_0, \ldots, x_n]$ to be

$$\frac{1}{n+1}(x_0 + \cdots + x_n).$$

Starting with a simplex Δ in $\mathbb{R}^N$ we obtain a simplicial complex Δ^1 by the following process, called *barycentric subdivision.* First we add a vertex $\hat{A}$ at the barycentre of each face A of Δ. Then we have a simplex through each set of vertices $\hat{A}_0, \ldots, \hat{A}_k$ if and only if after some reordering of these vertices we have

$$A_0 \subset A_1 \subset \cdots \subset A_k.$$

The *barycentric subdivision* of a simplicial complex X is defined to be the complex obtained by the barycentric subdivision of all the simplices in X. Clearly the geometric realisation of the barycentric subdivision of X is the same as that of X.

The *mesh* of a simplicial complex in $\mathbb{R}^N$ is the maximum of the diameters of its simplices. For any simplicial complex X, the mesh of the barycentric subdivision X^1 is at most $n/(n+1)$ times the mesh of X. Thus by repeatedly subdividing any simplicial complex we may make its mesh arbitrarily small.

Simplicial maps and simplicial approximations

For simplicial complexes X and Y, A *simplicial map* $s : X \to Y$ is a function from $|X|$ to $|Y|$ which takes simplices of X linearly onto simplices of Y. Any simplicial map is continuous (by the gluing lemma) and is completely determined by the images of the vertices in X, which are vertices in Y.

Each point $x \in |X|$ lies in the interior of a unique simplex in X, called the *carrier* of x. A simplicial map $s : X \to Y$ is a *simplicial approximation* of a continuous map $f : |X| \to |Y|$ if $s(x)$ lies in the carrier of $f(x)$ for each point $x \in |X|$. Note that if s is a simplicial approximation for f then s and f are homotopic, by a straight line homotopy.

The simplicial approximation theorem

For any continuous map $f : |X| \to |Y|$ there is an $m \geq 0$ such that there exists a simplicial approximation $s : X^m \to Y$ of f.

Proof

Let x be a vertex in X. Define the *open star* of x to be the union of the interiors of all the simplices in X which contain x. It is easy to prove that vertices $x_0, \ldots, x_k$ of X span a simplex if and only if the intersection of their open stars is non-empty.

First we prove the theorem in a special case. Suppose that for every vertex x in X there exists a vertex y in Y such that the open star of x is mapped by f into the open star of y. Then for every vertex x in X, choose such a y and define $s(x) = y$. By the above observation it follows that for any simplex $[x_0, \ldots, x_k]$ in X, the images $s(x_0), \ldots, s(x_k)$ span a simplex in Y, and hence we can extend our definition of s linearly over X to get a simplicial map $X \to Y$.

Now suppose that $x \in |X|$. We need to show that $s(x)$ lies in the carrier of $f(x)$. Let $[x_0, \ldots, x_k]$ be the carrier of x. Then x is in the intersection of the open stars of the x_i and so, by our assumption, $f(x)$ is in the intersection of the open stars of the $s(x_i)$. Therefore $s(x_0), \ldots, s(x_k)$ span a face of the carrier of $f(x)$, so the carrier of $f(x)$ contains each of the $s(x_i)$, and so it contains $s(x)$. Thus s is a simplicial approximation to f.

Now let f be an arbitrary map from X to Y. We shall show that for some $m \geq 0$, X^m satisfies the above condition. Now $|Y|$ is the union of the open stars U_y of its vertices, so the sets $f^{-1}(U_y)$ form an open cover of $|X|$. Since X is a compact metric space, by Lebesgue's lemma there exists $\epsilon > 0$ such that any open subset of $|X|$ of diameter less than ϵ is contained in one of the sets $f^{-1}(U_y)$. But for m sufficiently large, X^m has mesh less that ϵ — and so X^m satisfies the required condition.

Simplicial Homology

Orientations

Let Δ be an n-simplex with vertices $x_1, \dots, x_{n+1}$. If $n \geq 0$ then an *orientation* of Δ is an equivalence class of orderings of the x_i, where the orderings $x_1, \dots, x_{n+1}$ and $x_{\sigma(1)}, \dots, x_{\sigma(n+1)}$ are equivalent if and only if $\sigma \in A_{n+1}$. If $n = 0$ then an orientation of Δ is either 1 or -1.

An *oriented simplex* σ is a simplex together with a choice of orientation. We write $-\sigma$ to mean the simplex with the opposite choice of orientation. We shall write $[x_0, \dots, x_n]$ to mean the oriented simplex with the orientation corresponding to the given ordering of the vertices.

An oriented simplex induces an orientation on each of its codimension-one faces in the following manner:

- if i is even, the induced orientation of $[x_0, \dots, \hat{x}_i, \dots, x_n]$ is the one corresponding to this ordering of the vertices, and
- if i is odd, the induced orientation is the opposite to the one corresponding to this ordering.

We can check that this is well-defined.

Chains and boundaries

Let X be a simplicial complex. We define the group $C_k(X)$ of *k-chains* on X to be the free abelian group generated by the oriented k-simplices in X, modulo the relation that $(-1)\sigma = -\sigma$ for any oriented k-simplex σ.

For any oriented k-simplex σ in X, we define the *boundary* $\partial\sigma$ of σ to be the sum of the codimension-one faces of σ, with the orientations induced from σ. So $\partial\sigma \in C_{k-1}(X)$. We can extend this definition to get a group homomorphism $\partial : C_k(X) \to C_{k-1}$ in the obvious manner.

So we get the sequence

$$\cdots \xrightarrow{\partial} C_{k+1}(X) \xrightarrow{\partial} C_k(X) \xrightarrow{\partial} C_{k-1}(X) \xrightarrow{\partial} \cdots$$

In fact, the composition of any two of the consecutive homomorphisms is the zero homomorphism, or in other words, $\partial^2 = 0$. It suffices to prove that $\partial^2\sigma = 0$ for any oriented $(k+1)$-simplex σ. We have

$$\begin{aligned}
\partial^2[x_0, \dots, x_{k+1}] &= \partial \sum_{i=0}^{k+1} (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_{k+1}] \\
&= \sum_{i=0}^{k+1} (-1)^i \sum_{j=0}^{i-1} (-1)^j [x_0, \dots, \hat{x}_j, \dots, \hat{x}_i, \dots, x_{k+1}] \\
&\quad + \sum_{i=0}^{k+1} (-1)^i \sum_{j=i+1}^{k+1} (-1)^{j-1} [x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}],
\end{aligned}$$

where the terms in the final expression cancel in pairs.

Homology groups

Define the subgroup $Z_k(X) \leq C_k(X)$ of *k-chains* in X to be the kernel of ∂, and define the subgroup $B_k(X) \leq C_k(X)$ of *k-boundaries* to be the image of ∂. Then since $\partial^2 = 0$, $B_k(X)$ is contained within $Z_k(X)$ and so we may define the *kth homology group* of X as the quotient

$$H_k(X) = Z_k(X)/B_k(X).$$

It is easy to see that $H_0(X)$ is a free abelian group whose rank is the number of connected components of X.

Homology groups of an n-simplex

If X is a point, then by considering the groups of k-chains on X we see immediately that $H_0(X) = \mathbb{Z}$ and $H_i(X) = 0$ for $i \geq 1$.

To compute the homology groups of an n-simplex for $n \geq 2$ we may use the following device. If $X \subset \mathbb{R}^N$ is a simplicial complex, define the *cone* on X to be the simplicial complex CX formed by embedding $\mathbb{R}^N$ in $\mathbb{R}^{N+1}$, choosing a point $v \in \mathbb{R}^{N+1} \setminus \mathbb{R}^N$ and then taking the union of all the line segments from v to each of the points in X. Then we see that CX is a simplicial complex, whose simplices are those in X, those formed as the convex hull of v and a simplex in X, and the point v itself.

The homology groups of any cone are the same as those of a point. To prove this we define the homomorphism $d : C_q(CX) \to C_{q+1}(CX)$ which sends an oriented q-simplex $\sigma = [v_0, \ldots, v_q]$ in CX to $[v, v_0, \ldots, v_q]$ if σ is contained in X, or 0 otherwise. We can check that this is a well-defined homomorphism. Then we show that

$$(\partial \circ d)(\sigma) = \sigma - (d \circ \partial)(\sigma)$$

for any oriented q-simplex σ with $q \geq 1$. But then if z is any q-cycle in CX with $q \geq 1$,

$$\partial(d(z)) = z - d(\partial(z)) = z - d(0) = z$$

and so z is a boundary. Hence $H_q(CX) = 0$.

Now since an $(n+1)$-simplex is just the cone on an n-simplex, we have that for any simplex X, $H_0(X) = \mathbb{Z}$ and $H_i(X) = 0$ for $i \geq 1$.

Homology groups of S^n

If $n = 0$ then the homology groups of S^n are clearly $H_0(S^0) = Z^2$, $H_i(S^0) = 0$ for $i \geq 1$.

If $n \geq 1$ then we may triangulate S^n is a simple way. Let Δ denote the simplicial complex which is the union of all the faces of an $(n+1)$-simplex, and let Σ denote the same simplicial complex but without the single $(n+1)$-dimensional face. Then Σ is a triangulation of S^n. Now the chain complex of Σ is exactly the same as that of Δ, except that we have $C_{n+1}(\Sigma) = 0$, whereas $C_{n+1}(\Delta) \cong \mathbb{Z}$, generated by the single $(n+1)$-dimensional face.

Therefore, for $i \leq (n-1)$ the homology groups $H_i(\Delta)$ and $H_i(\Sigma)$ are the same. But then we see that $H_n(\Sigma)$ is the kernel of the map $\partial : C_n(\Sigma) \to C_{n-1}(\Sigma)$, which is also the kernel of $\partial : C_n(\Delta) \to C_{n-1}(\Delta)$. But by the exactness of the sequence for Δ, this is just the image of $\partial : C_{n+1}(\Delta) \to C_n(\Delta)$, which is isomorphic to $C_{n+1}(\Delta) \cong \mathbb{Z}$ as ∂ is injective.

We will show that the homology groups of a space are independent of its triangulation, and hence we may conlude that

$$H_i(S^n) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0 \text{ and } i = 0 \\ \mathbb{Z} & \text{if } n \geq 1 \text{ and } i = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

Induced homomorphisms

Let $f : X \to Y$ be a simplicial map. Then f determines a homomorphism

$$f_* : C_i(X) \longrightarrow C_i(Y)$$

by

$$f_*([x_0, \ldots, x_i]) = \begin{cases} [f(x_0), \ldots, f(x_i)] & \text{if } f(x_0), \ldots, f(x_i) \text{ are all distinct} \\ 0 & \text{otherwise.} \end{cases}$$

We can check that f_* is a chain map, that is, that

$$\partial \circ f_* = f_* \circ \partial$$

and so the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial} & C_{i+1}(X) & \xrightarrow{\partial} & C_i(X) & \xrightarrow{\partial} & C_{i-1}(X) & \xrightarrow{\partial} & \cdots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ \cdots & \xrightarrow{\partial} & C_{i+1}(Y) & \xrightarrow{\partial} & C_i(Y) & \xrightarrow{\partial} & C_{i-1}(Y) & \xrightarrow{\partial} & \cdots \end{array}$$

It follows that f_* maps $Z_i(X)$ into $Z_i(Y)$, and $B_i(X)$ into $B_i(Y)$, and so f_* gives rise to a homomorphism, also denoted f_*, from $H_i(X)$ to $H_i(Y)$.

Now it can be shown that barycentric subdivision of a complex does not change the homology groups (proof omitted — see Armstrong pp. 185–188). Thus if $f : |X| \to |Y|$ is any continuous map then we may define a homomorphism

$$f_* : H_i(X) \to H_i(Y)$$

as the composition

$$H_i(X) \cong H_i(X^m) \xrightarrow{s_*} H_i(Y),$$

where $s : X \to Y$ is some simplicial approximation to f. It can be shown that this is well-defined, independent of the choice of s, by showing that any two "close" simplicial maps give rise to the same homomorphisms of homology groups.

Functorial properties. Thus homology groups are invariant under homotopy equivalence.

Applications

S^m and S^n are not homotopy equivalent and thus $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$. The Brouwer fixed-point theorem.

The Mayer–Vietoris Sequence

Let X be a simplicial complex which is the union $A \cup B$ of two subcomplexes. Then

$$\cdots \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(X) \longrightarrow H_{i-1}(A \cap B) \longrightarrow \cdots$$

is a long exact sequence. What are the three homomorphisms?

- The homomorphism $H_i(A \cap B) \to H_i(A) \oplus H_i(B)$ is the obvious pair of inclusions.
- The homomorphism $H_i(A) \oplus H_i(B) \to H_i(X)$ is given by $(x, y) \mapsto x - y$.
- The homomorphism $H_i(X) \to H_{i-1}(A \cap B)$ is the "boundary map" constructed as follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_i(A \cap B) & \longrightarrow & C_i(A) \oplus C_i(B) & \longrightarrow & C_i(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \longrightarrow & C_{i-1}(A \cap B) & \longrightarrow & C_{i-1}(A) \oplus C_i(B) & \longrightarrow & C_{i-1}(X) & \longrightarrow & 0
\end{array}$$

For any element of $H_i(X)$, take a representative cycle $z \in Z_i(X)$ and choose an element $d \in C_i(A) \oplus C_i(B)$ which maps to z. Then the equivalence class of z maps to the equivalence class of the (unique) inverse image $c \in Z_{i-1}(A \cap B)$ for $\partial(d)$.