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CODIMENSION ONE HOLOMORPHIC FOLIATIONS ON $\mathbb{P}^n_{\mathbb{C}}$: PROBLEMS IN COMPLEX GEOMETRY.

by

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En l'honneur de H. Hironaka pour son 80$^{\text{ème}}$ *anniversaire*

Abstract. — After a short review on foliations, we prove that a codimension 1 holomorphic foliation on $\mathbb{P}^3_{\mathbb{C}}$ with simple singularities is given by a closed rational 1-form. The proof uses Hironaka-Matsumura prolongation theorem of formal objects.

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Preliminaries

Let M be a complex compact connected manifold of dimension $n \geq 2$. A codimension 1 singular holomorphic foliation $\mathcal{F}$ on M is given by a covering by open subsets $(V_j)_{j\in J}$ and a collection of integrable holomorphic 1-forms ω_j on V_j, $\omega_j \wedge d\omega_j = 0$, having codimension ≥ 2 zero sets such that on each non empty intersection $V_j \cap V_k$:

(*) $\quad \omega_j = g_{jk}.\omega_k$ with $g_{jk} \in \mathcal{O}^*(V_j \cap U_k)$.

Let $\operatorname{Sing}\omega_j := \{p \in V_j \ , \ \omega_j(p) = 0\}$ be the singular set of ω_j. Condition (*) implies that $\operatorname{Sing}\mathcal{F} := \cup_{j\in J} \operatorname{Sing}\omega_j$ is a codimension ≥ 2 analytic subset of M, the singular set of $\mathcal{F}$.

In the special case where M is a projective manifold and $\mathcal{F}$ a foliation as above, we can associate to $\mathcal{F}$ a meromorphic 1-form ω in the following way. We take a rational vector fields Z on M, not tangent to $\mathcal{F}$, that is $h_j = i_{Z_{|U_j}}\omega_j \not\equiv 0$; the meromorphic 1-form ω defined on V_j by $\omega_{|U_j} = \omega_j/h_j$ is global and integrable. In this case we will say that ω defines $\mathcal{F}$.

There is another interesting very special case: the case $M = \mathbb{P}^n_{\mathbb{C}}$, the n dimensional complex projective space. In that context, we have a theorem of Chow-type. Denote by $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$ the natural projection, and consider $\pi^{-1}\mathcal{F}$ the pull-back of $\mathcal{F}$ by π; with the previous notations, $\pi^{-1}\mathcal{F}$ is defined by the 1-form $\pi^*\omega_j$ on $\pi^{-1}(U_j)$. Recall that, for $n \geq 2$, we have $H^1(\mathbb{C}^{n+1} \setminus \{0\}, \mathcal{O}^*) = \{1\}$: it is a result due to Cartan ([8]). As a consequence, there exists a global holomorphic 1-form ω on $\mathbb{C}^{n-1} \setminus \{0\}$ which defines $\pi^{-1}\mathcal{F}$ on $\mathbb{C}^{n+1} \setminus \{0\}$.

By Hartog's prolongation theorem ω can be extended holomorphically at 0. By construction we have $i_R\omega = 0$, where R is the Euler (or radial) vector fields:

$$R = \sum_{i=0}^{n} z_j \frac{\partial}{\partial z_j}.$$

This fact and the integrability condition imply that ω is colinear to an integrable homogeneous 1-form $\omega_{\nu+1} = \sum_{i=0}^{n} A_i(z)dz_i$, A_i homogenenous polynomials of degree $\nu+1$, $\gcd(A_0,\cdots,A_n) = 1$ (*i.e.* $\operatorname{cod}\operatorname{Sing}\omega_{\nu+1} \geq 2$). This is the so-called foliated Chow's Thoerem:

Theorem 0.1. — *To any codimension* 1 *holomorphic foliation* $\mathcal{F}$ *on* $\mathbb{P}^n_{\mathbb{C}}$ *is asociatied an homogeneous integrable* 1*-form* $\omega_{\nu+1}$ *on* $\mathbb{C}^{n+1}$ *defining* $\pi^{-1}\mathcal{F}$ *with* $\operatorname{cod}\operatorname{Sing}\omega_{\nu+1} \geq 2$.

By definition the integer ν is the degree of the foliation $\mathcal{F}$. The homogeneous 1-form $\omega_{\nu+1}$ is well defined up to multiplication by non zero complex number.

Remark 0.2. — Denote by $U_i := \{z_i = 1\} \subset \mathbb{P}^n_{\mathbb{C}}$ the usual affine charts associated to the projective coordinates $(z_0 : \cdots : z_n)$. Then $\omega_{\nu+1_{|U_j}} = \omega_j$ is a polynomial 1-form on $U_j \simeq \mathbb{C}^n$ which can be extended meromorphically to $\mathbb{P}^n_{\mathbb{C}}$.

We have the following facts; if $\mathcal{F}$ is a foliation of degree ν on $\mathbb{P}^n_{\mathbb{C}}$ then:

- the integer $\nu = \deg\mathcal{F}$ is exactly the number of tangencies of $\mathcal{F}$ with a generic line L, that is the number of points $m \in L$ where L is not transverse to $\mathcal{F}$ (if $m \in U_j$ then L is "contained" in the kernel of the linear form $\omega_j(m)$).
- the set $\operatorname{Sing}\mathcal{F}$ has non trivial components of codimension 2: points in $\mathbb{P}^2_{\mathbb{C}}$, curves in $\mathbb{P}^3_{\mathbb{C}}\ldots$ In particular, there are no non singular codimension 1 foliations on $\mathbb{P}^n_{\mathbb{C}}$ except for $n = 1$. This fact can be proved by using De Rham-Saito division lemma [18].

If $\mathcal{F}$ is a codimension 1 foliation on M, the leaves of $\mathcal{F}$ are, by defintion, the leaves (maximal integral immersed manifolds) of the regular foliation $\mathcal{F}_{|M\setminus\operatorname{Sing}\mathcal{F}}$.

An exciting problem is to give the description of the spaces $\mathcal{F}(n;d)$ of codimension 1 foliations of degree d on $\mathbb{P}^n_{\mathbb{C}}$, in particular the irreducible components of these spaces. For $n = 2$ the sets $\mathcal{F}(2;d)$ are Zariski open sets in some projective spaces and the consistency of the problem appears in dimension ≥ 3.

A second problem consists, for each given irreducible component of $\mathcal{F}(n;d)$, in the description of the leaves of generic elements of that component.

1. Some examples and known facts.

There are many examples of foliations without singularities, in particular on tori, Hopf manifolds etc. Regular foliations on compact complex surfaces are classified by Brunella ([1]). Here we focus on foliations in $\mathbb{P}^n_{\mathbb{C}}$. As we have seen above, such foliations are singular.

Example 1.1. — Foliations of degree 0 on $\mathbb{P}^n_{\mathbb{C}}$.

Such foliations are pencils of hyperplanes. Up to conjugacy by $\operatorname{Aut}\mathbb{P}^n_{\mathbb{C}}$, the group of automorphisms of $\mathbb{P}^n_{\mathbb{C}}$, there is one model, the foliation $\mathcal{F}_0$ given by the homogeneous 1-form $z_0dz_1 - z_1dz_0$. Note that $\mathcal{F}_0$ is also given by the global closed 1-form $\frac{dz_0}{z_0} - \frac{dz_1}{z_1}$. The singular locus of $\mathcal{F}_0$ is the linear space $\{z_0 = z_1 = 0\} \simeq \mathbb{P}^{n-2}_{\mathbb{C}}$ and the closure of the leaves are hyperplanes $z_0/z_1 =$cste. Remark also that, by blowing-up the singular locus, we obtain a regular foliation on the blow-up of $\mathbb{P}^n_{\mathbb{C}}$.

The space $\mathcal{F}(n;0)$ is isomorphic to the Grassmanian of $(n-2)$-linear subspaces of $\mathbb{P}^n_{\mathbb{C}}$.

Example 1.2. — Foliations of degree 1 on $\mathbb{P}^2_{\mathbb{C}}$.

A generic element of $\mathcal{F}(2;1)$ is given in a good chart $\{(x,y)\} \simeq \mathbb{C}^2$ by the linear 1-form $\lambda y dx - x dy$, $\lambda \in \mathbb{C}$. The leaves are parametrized by

$$\mathbb{C} \ni t \mapsto (x_0 e^t, y_0 e^{\lambda t}) \in \mathbb{C}^2 \subset \mathbb{P}^2_{\mathbb{C}}.$$

If $\lambda \in \mathbb{Q}$, the closure of the leaves are rational algebraic curves (of type $x^p y^q$ =cste), and if $\lambda \in \mathbb{C} \setminus \mathbb{Q}$ the leaves are transcendental Pfaffian sets.

All foliations of degree 1 on $\mathbb{P}^2_{\mathbb{C}}$ are given by a closed rational 1-form ($\lambda \frac{dx}{x} - \frac{dy}{y}$ in the generic case). The set $\mathcal{F}(2;1)$ can be identified to a Zariski open set in the projective space $\mathbb{P}^7_{\mathbb{C}}$.

Example 1.3. — The set $\mathcal{F}(n;1)$, $n \geq 3$.

For $n \geq 3$, the set $\mathcal{F}(n;1)$ has two irreductible components corresponding to the following alernative; if $\mathcal{F} \in \mathcal{F}(n;1)$:

(*) either there exists a linear map $F : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ and $\mathcal{F}_0 \in \mathcal{F}(2;1)$ such that $\mathcal{F} = F^{-1}\mathcal{F}_0$.

(**) or in a good affine chart $\mathbb{C}^n \subset \mathbb{P}^n_{\mathbb{C}}$, $\mathcal{F}$ is given by the 1-form $\omega = dP$, where P is polynomial of degree 2. The leaves are the level sets of P.

In each of these two cases $\mathcal{F}$ is given by a closed rational 1-form.

Example 1.4. — Quadratic foliations on $\mathbb{P}^n_{\mathbb{C}}$, $n \geq 3$.

The description of $\mathcal{F}(n;2)$, $n \geq 3$, is a little bit more difficult; $\mathcal{F}(n;2)$ has six irreductible components ([**9**]) and we have the following alternative. If $\mathcal{F} \in \mathcal{F}(n;2)$, $n \geq 3$, then:

(*) either there exists a linear map $F : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ and a foliation $\mathcal{F}_0 \in \mathcal{F}(2;2)$ on $\mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{F} = F^{-1}\mathcal{F}_0$, the pull-back of $\mathcal{F}_0$ by F (it corresponds to one component of $\mathcal{F}(n;2)$).

(**) or $\mathcal{F}$ is defined by a closed rational 1-form. This second part of the alternative gives 5 components.

One of the component is a $\mathrm{Aut}\,\mathbb{P}^n_{\mathbb{C}}$-orbit of the so-called "exceptional foliation"; this means in particular that there exists quadratic stable foliations. In fact for any d the set $\mathcal{F}(n;d)$, $n \geq 3$, contains stable foliations ([**3**]).

Example 1.5. — Foliations associated to closed meromorphic 1-forms.

To each meromorphic closed 1-form ω on $\mathbb{P}^n_{\mathbb{C}}$ is associated a codimension 1-holomorphic foliation. Recall that such a closed form has a decomposition:

$$\omega = \Sigma \lambda_i \frac{d f_i}{f_i} + dh$$

where the $\lambda_i' s$ are complex numbers (the residues or periods) and the $f_i' s$ and h are rational functions. The leaves are (outside the singular set of the foliation) the connected components of the "level sets" of the multivalued function $\Sigma \lambda_i \log f_i + h$. There are many deep questions concerning the nature of these leaves in relation with topology, number theory, hyperbolic geometry....

As it can be seen in [**2**], [**15**], for each degree d, these are several irreducible components of $\mathcal{F}(n;d)$, $n \geq 3$, whose generic elements correspond to foliations given by closed 1-forms.

Example 1.6. — Degree 3 foliations.

The explicit decomposition in irreducible components of the space $\mathcal{F}(n;3)$, $n \geq 3$, is not known. Nevertheless there is a qualitative description of the elements of $\mathcal{F}(n;3)$; in fact we have an alternative quasi-similar to Example 1.4: for $\mathcal{F} \in \mathcal{F}(n;3)$

(*) either there exist $F : \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ rational and $\mathcal{F}_0$ a foliation on $\mathbb{P}^2_{\mathbb{C}}$ such that $\mathcal{F} = F^{-1}\mathcal{F}_0$,

(**) or $\mathcal{F}$ is defined by a closed rational 1-form.

This alternative is the consequence of the two papers [**10**] and [**16**]; the difference with Example 1.4 is that there is no control of the degrees of the rational map F and the foliation $\mathcal{F}_0$.

There exist foliations in degree >3 on $\mathbb{P}^n_{\mathbb{C}}$, $n\geq 3$, which don't satisfy the alternative of Example 1.6. We will now speak a little bit of families of such examples, the so-called transversally projective foliations.

Example 1.7. — Transversally projective foliations.

To such a foliation $\mathcal{F}_0$ is associated a "$\mathrm{sl}(2;\mathbb{C})$ 3-uple" $(\omega_0,\omega_1,\omega_2)$ of rational 1-forms on $\mathbb{P}^n_{\mathbb{C}}$ satisfying:
(*) $\mathcal{F}_0$ is given by ω_0,
(**) the ω_i verify the Maurer-Cartan conditions:

$$d\omega_0=\omega_0\wedge\omega_1,\qquad d\omega_1=\omega_0\wedge\omega_2,\qquad d\omega_2=\omega_1\wedge\omega_2.$$

Remark that a foliation given by a closed rational 1-form ω_0 is a special case of transversally projective foliation (take $\omega_1=\omega_2=0$). The Maurer-Cartan conditions imply the integrability of the unfolding:

$$\Omega=dt+\omega_0+t\omega_1+\tfrac{t^2}{2}\omega_2,\quad t\in\mathbb{C}\subset\mathbb{P}^1_{\mathbb{C}},$$

which defines a "Riccati-foliation" on $\mathbb{P}^n_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$. We see that the restriction Ω to $t=0$ gives the foliation $\mathcal{F}_0$ and the restriction to $t=\infty$ gives, in the case $\omega_2\not\equiv 0$, a new foliation $\mathcal{F}_2$ associated to ω_2.

Note also that to an ordinary Riccati differential equation $\frac{dy}{dx}=a(x)y^2+b(x)y+c(x)$, $a,b,c\in\mathbb{C}(x)$ is associated a transversally projective foliation on $\mathbb{P}^3_{\mathbb{C}}$ given (in an affine chart) by:

$$\omega_0=dz+\omega_0'+z\omega_1'+\frac{z^2}{2}\omega_2'$$

with, denoting by L the Lie derivative:

$$\omega_0'=dy-(a(x)y^2+b(x)y+c(x))dx,\qquad \omega_1'=L_{\frac{\partial}{\partial y}}\omega_0',\qquad \omega_2'=L_{\frac{\partial}{\partial y}}\omega_1'.$$

Here the corresponding $\mathrm{sl}(2;\mathbb{C})$ 3-uple is $(\omega_0,\omega_1=\omega_1'+z\omega_2',\omega_2=\omega_2')$. We have the following fact: there are explicit constructions of transversally projective foliations on $\mathbb{P}^3_{\mathbb{C}}$ associated to some special rational Hilbert-modular surfaces ([**11**]). These foliations are not defined by closed meromorphic 1-forms and are not rational pull-back of foliations on $\mathbb{P}^2_{\mathbb{C}}$ (*see* [**11**]).

All known foliations $\mathcal{F}$ on $\mathbb{P}^n_{\mathbb{C}}$, $n\geq 3$, satisfy the following alternative I: $\mathcal{F}$ is
(*) either transversally projective
(**) or a rational pull-back of a foliation $\mathcal{F}_0$ on $\mathbb{P}^2_{\mathbb{C}}$.

We don't know if the previous alternative I is always satisfied or if there exist other types of foliations on $\mathbb{P}^n_{\mathbb{C}}$.

It is possible to prove that a transversally projective foliation $\mathcal{F}$ on $\mathbb{P}^n_{\mathbb{C}}$ has an invariant hypersurface $X\subset\mathbb{P}^n_{\mathbb{C}}$ (*see* [**11**]): $X\setminus\mathrm{Sing}\mathcal{F}$ is a leaf of the regular foliation $\mathcal{F}_{|\mathbb{P}^n_{\mathbb{C}}\setminus\mathrm{Sing}\mathcal{F}}$. For example if $\mathcal{F}$ is given by a closed 1-form ω, then the divisor of the poles of ω is such an invariant hypersurface.

A contrario, generic foliations on $\mathbb{P}^2_{\mathbb{C}}$, with degree ≥ 2, have no invariant algebraic curves ([**14**]). This implies that general pull-back foliations $F^{-1}\mathcal{F}_0$, $F:\mathbb{P}^n_{\mathbb{C}}\dashrightarrow\mathbb{P}^2_{\mathbb{C}}$, don't have invariant hypersurfaces.

The following conjecture due to Brunella says that a foliation on $\mathbb{P}^n_{\mathbb{C}}$ either is a rational pull-back of a foliation on $\mathbb{P}^2$ or has an invariant algebraic hypersurface (alternative II). As we have seen alternative I implies alternative II and alternative I is satisfied in small degree (≤ 3) for foliations on $\mathbb{P}^n_{\mathbb{C}}$. We mention that alernative I is always satisfied for foliations on $\mathbb{P}^n_{\Bbbk}$, where $\Bbbk$ is a field of positive characteristic ([**11**]).

2. Reduction of singularities for codimension one foliations in dimension ≤ 3.

In dimension 2, Seidenberg gives in [**19**] the first statement concerning the reduction of singularities. For germs of analytic subsets X in $\mathbb{C}^n,0$ we know, following Hironaka, that after suitable blow-up $\pi\colon\widetilde{\mathbb{C}}^n\to\mathbb{C}^n,0$ the total transform $\pi^{-1}(X)$ is locally given by the zeroes of an ideal generated by "monomials". For foliations, due to the divergence of certain normal forms, the local models after reduction of singularities are formal one. Seidenberg's result was generalized in dimension 3 by Cano-Cerveau ([6], non-dicritical case) and Cano ([5], general case). This reduction of singularities for foliation allows to prove Thom's conjecture about invariant local hypersurfaces, in dimension 2 by Camacho-Sad ([4]), in dimension 3 by Cano-Cerveau ([6]) and in any dimension by Cano-Mattei ([7]):

Theorem 2.1. — *Any codimension 1 germ of non dicritical holomorphic foliation has an invariant hypersurface.*

Recall that there exist, in the dicritical case, codimension 1 holomorphic foliations without local invariant hypersurface ([14]).

We give now the precise statement of reduction of singularities in dimension 3 (an adapted version to a divisor is given in [5] and [6]).

Theorem 2.2. — *Let $\mathcal{F}$ be a codimension 1 holomorphic foliation over $X = \mathbb{C}^3, 0$. Then there is a finite sequence of permissible blow-ups:*

$$X = X(1) \xleftarrow{\pi(1)} X(2) \xleftarrow{\pi(2)} \cdots \xleftarrow{\pi(N)} X(N)$$

such that at each point $x \in X(N)$ the strict transform $\mathcal{F}_N$ of $\mathcal{F}$ by $\pi(N)\circ\cdots\circ\pi(1)$ either is non singular or has simple singularity.

The description of simple singularities can be given in terms of convenient adapted multiplicities ([5], [6]). One of the difficulties in the theory is to give local models (like monomial equations in the case of hypersurfaces) for these simple singularities. After that, we can think that the simple singularities are given by these normal forms.

So, let $\mathcal{F}$ be a codimension 1 holomorphic foliation over $\mathbb{C}^3, 0$. The foliation is said to be simple ([6]) if and only if there exists a formal diffeomorphism $\phi \in \widehat{\mathrm{Diff}(\mathbb{C}^3,0)}$ (the formal completion of the group $\mathrm{Diff}(\mathbb{C}^3,0)$ of germs of holomorphic diffeomorphisms) such that $\phi^{-1}\mathcal{F}$ is given by one of the following meromorphic 1-forms:

(1) $\frac{dx}{x} + \lambda\frac{dy}{y}$, $\lambda \in \mathbb{C}\setminus\mathbb{Q}_-$,

(2) $\frac{dx}{x} + \left(\varepsilon + \frac{1}{y^s}\right)\frac{dy}{y}$, $s \in \mathbb{N}\setminus\{0\}$, $\varepsilon \in \mathbb{C}$,

(3) $\frac{dx}{x} + \left(\varepsilon + \frac{1}{(x^p y^q)^s}\right)\left(p\frac{dx}{x} + q\frac{dy}{y}\right)$, $\gcd(p,q) = 1$, $s \in \mathbb{N}\setminus\{0\}$, $\varepsilon \in \mathbb{C}$.

(4) $\alpha\frac{dx}{x} + \beta\frac{dy}{y} + \frac{dz}{z}$, $\alpha\beta \neq 0$, $\alpha, \beta, \alpha/\beta \in \mathbb{C}\setminus\mathbb{Q}_-$,

(5) $\frac{dx}{x} + \beta\frac{dy}{y} + (\varepsilon + \frac{1}{z^s})dz$, $s \in \mathbb{N}\setminus\{0\}$, $0 \neq \beta \in \mathbb{C}\setminus\mathbb{Q}_-$

(6) $\frac{dx}{x} + \beta\frac{dy}{y} + \left(\varepsilon + \frac{1}{(y^q z^q)^s}\right)\left(p\frac{dy}{y} + q\frac{dz}{z}\right)$, $s \in \mathbb{N}\setminus\{0\}$, $gcd(p,q) = 1$, $\varepsilon, \beta \in \mathbb{C}$

(7) $\frac{dx}{x} + \beta\frac{dy}{y} + \left(\varepsilon + \frac{1}{(x^p y^q z^r)^s}\right)\left(p\frac{dx}{x} + q\frac{dy}{y} + r\frac{dz}{z}\right)$, $s \in \mathbb{N}\setminus\{0\}$, $\gcd(p,q,r) = 1$, ε, $\beta \in \mathbb{C}$;

x, y, z are linear coordinates in $\mathbb{C}^3$.

In some sense the seven types of previous 1-forms describe the normal forms of generic meromorphic 1-forms with normal crossing divisors of poles. Note that the forms (1), (2) and (3) are the models in dimension 2. Remark also that if $\mathcal{F}$ is a foliation over M with only simple singularities, then $\mathcal{F}$ is given at each point $x \in M$ by a (formal) meromorphic 1-form Ω_x.

The previous 1-form Ω_x is unique (up to multiplication by a complex number) except when $\mathcal{F}_{,x}$ has a non constant holomorphic first integral. The reason is that if two (formal) meromorphic closed 1-forms Ω_1 and Ω_2 give the same foliation, then $\Omega_2 = f.\Omega_1$ where f is a (formal) meromorphic function. By differentiation, we see that, if f is non-constant, f is a (formal) meromorphic first integral. But it is easy to see that if $\mathcal{F}_{,x}$ is either non singular or simple, and if $\mathcal{F}_{,x}$ has a (formal) non constant first integral, then $\mathcal{F}_{,x}$ has a ordinary (formal) first integral (without poles); following Malgrange's singular Frobenius theorem $\mathcal{F}_{,x}$ has a non constant holomorphic first integral ([17]).

3. Foliations with simple singularities on $\mathbb{P}^3_{\mathbb{C}}$

As we have seen, any codimension 1 holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^3_{\mathbb{C}}$ has a non trivial curve of singularities. We consider now special foliations on $\mathbb{P}^3_{\mathbb{C}}$.

Proposition 3.1. — *Let $\mathcal{F}$ be a codimension 1 holomorphic foliation on $\mathbb{P}^3_{\mathbb{C}}$. Suppose that there exists a component γ (of dimension 1) of the singular locus* Sing $\mathcal{F}$ *such that:*

1) for any $x \in \gamma$, the germ $\mathcal{F}_{,x}$ has a simple singularity, in other words $\mathcal{F}$ is reduced along γ;

2) for any $x \in \gamma$, the germ $\mathcal{F}_{,x}$ has not a holomorphic non constant first integral.

Then $\mathcal{F}$ is given by a global closed meromorphic 1-form. In particular $\mathcal{F}$ is transversally projective.

Proof. — Assume at first that all the local models of $\mathcal{F}$ along γ are given by meromorphic closed one-forms (that is the normalizing diffeomorphisms ϕ are convergent one). Then there exist a finite covering of γ by open sets U_α and closed meromorphic 1-forms Ω_α defined on U_α such that $\mathcal{F}_{|U_\alpha}$ is given by Ω_α. If ω is a global rational 1-form on $\mathbb{P}^3_{\mathbb{C}}$ associated to $\mathcal{F}$ we have: $\Omega_\alpha = H_\alpha.\omega_{|U_\alpha}$, with H_α meromorphic on U_α. On a non trivial intersection $U_\alpha \cap U_\beta$ we have:

$$\Omega_\alpha = \lambda_{\alpha\beta}\Omega_\beta.$$

By hypothesis for a good choice of the covering, the cocycles $\lambda_{\alpha\beta}$ are constant.

The equality $H_\alpha = \lambda_{\alpha\beta}H_\beta$ gives by differentiation $\frac{dH_\alpha}{H_\alpha} = \frac{dH_\beta}{H_\beta}$ and the $\frac{dH_\alpha}{H_\alpha}$ define a closed meromorphic 1-form ω_1' on a neighborhood of γ.

In fact, due to the possible divergence of the normalizing transformations, the local models $\Omega_x, x \in \gamma$, are not a priori convergent; a delicate study of the normalisations $\phi = \phi_x$ allows to study the dependence of Ω_x relative to $x \in \gamma$ and to see that the form ω_1' is a "formal meromorphic 1-form along γ". The deep works [12] and [13] say that the form ω_1' is in fact the restriction of a global closed meromorphic 1-form ω_1 on $\mathbb{P}^3_{\mathbb{C}}$. This form ω_1 has a decomposition as in Example 1.4:

$$\omega_1 = \Sigma\lambda_i\frac{df_i}{f_i} + dh$$

with $\lambda_i \in \mathbb{C}$, f_i and h rational functions.

Using the local construction of ω_1 ($\omega_{1,x} = \frac{dH_x}{H_x}$ for $x \in \gamma$) we see that $\lambda_i \in \mathbb{Z}$ and $h \equiv 0$; so $\omega_1 = \frac{dH}{H}$ for some rational function H. If we come back to the relations $\Omega_\alpha = H_\alpha.\omega_{|U_\alpha}$, we observe that

$$\frac{dH}{H} \wedge \omega + d\omega = 0$$

and the rational 1-form $H.\omega$ is closed and defines the foliation $\mathcal{F}$. □

Remark 3.2. — In [15] Lins Neto uses that idea to glue local meromorphic closed Pfaffian forms to obtain a proximate result in a particular case.

Now the next statement says that if a foliation $\mathcal{F}$ on $\mathbb{P}^3_{\mathbb{C}}$ has only simple singularities, then there exists a component γ of Sing $\mathcal{F}$ satisfying Proposition 3.1.

Proposition 3.3. — *Let $\mathcal{F}$ be a holomorphic codimension 1 foliation over $\mathbb{P}^3_{\mathbb{C}}$. Suppose that all the non isolated singularities of $\mathcal{F}$ are simple; then the hypothesis of Proposition 3.1 are satisfied.*

Proof. — Take a generic linear planar section $i\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ and denote by $\mathcal{F}_0$ the "restriction" $i^{-1}\mathcal{F}$. It can be seen that all the singular points of $\mathcal{F}_0$ are simple (in the sense of dimension 2). For such a singular point there is an index, the so-called Baum-Bott index. For a "hyperbolic" singular point m_0 of $\mathcal{F}_0$, that is given locally by a 1-form of type:

$$\lambda_1 x dy - \lambda_2 y dx + \cdots$$

the Baum-Bott index is by definition $\mathrm{BB}(\mathcal{F}_0; m_0) = \frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2}$. In the general case $\mathrm{BB}(\mathcal{F}_0; m)$ is given by an explicit integral formula ([1]). There is a global index formula relating the local $\mathrm{BB}(\mathcal{F}_0; m_0)$ to some special Chern-class, namely in the case of $\mathbb{P}^2_{\mathbb{C}}$:

$$\sum_{m_0 \in \mathrm{Sing}\,\mathcal{F}_0} \mathrm{BB}(\mathcal{F}_0; m_0) = (n+2)^2$$

where n is the degree of the foliation $\mathcal{F}_0$; this is the Baum-Bott formula ([1]).

Note that if $m_0 = i^{-1}(m)$ is a contact-singularity of $\mathcal{F}_0$, *i.e.* $m \notin \mathrm{Sing}\,\mathcal{F}$, then $\mathcal{F}_0$ has a local holomorphic first integral of Morse type at m_0; as a consequence we have $\mathrm{BB}(\mathcal{F}_0; m_0) = 0$. Suppose now that for all point x belonging to any dimension 1 component γ_i of $\mathrm{Sing}\,\mathcal{F}$, the germ $\mathcal{F}_{,x}, x \in \gamma_i$, has a local non trivial holomorphic first integral. Then at each point $m_0 \in \mathrm{Sing}\,\mathcal{F}_0$, the foliation is given by a 1-form of the following type:

$$\omega_{,m_0} = pydx + qxdy, \quad p, q \in \mathbb{N} \setminus \{0\},\ \gcd(p,q) = 1.$$

We see that the Baum-Bott index $\mathrm{BB}(\mathcal{F}_0; m_0)$ is negative, in contradiction with the Baum-Bott formula. □

We are now able to give the main result of this paper:

Theorem 3.4. — *Let $\mathcal{F}$ be a codimension 1 holomorphic foliation on $\mathbb{P}^3_{\mathbb{C}}$. Suppose that all the non isolated singularities of $\mathcal{F}$ are simple. Then $\mathcal{F}$ is given by a closed rational 1-form.*

A standard extension result implies the:

Corollary 3.5. — *Let $\mathcal{F}$ be a codimension 1 holomorphic foliation on $\mathbb{P}^n_{\mathbb{C}}, n \geq 3$. Suppose that there exists a linear section $i\colon \mathbb{P}^3_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$, such that $i^{-1}\mathcal{F}$ is like in Theorem 3.4. Then we have the same conclusion: $\mathcal{F}$ is given by a closed meromorphic 1-form.*

Remark 3.6. — The structure of the ambiant space is important. For example a generic foliation on $\mathbb{P}^2_{\mathbb{C}}$ has only simple sigularities and is not given by a closed 1-form; so there exist foliations on $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with only simple singularities which are not given by closed 1-forms.

Remark 3.7. — A consequence of Theorem 3.4 is the following: it is not possible to realize local dimension 3 simple singularities with divergent normalization, by a global foliation on $\mathbb{P}^3_{\mathbb{C}}$ having only simple singularities.

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