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Elliptic Functions and Elliptic Curves

PATRICK DU VAL

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PATRICK DU VAL

Ordinarius Professor of Geometry,
University of Istanbul

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Contents

Preface

The lectures on which the following notes are based were given in various forms in University College, London, from about 1964 to 1969. Generally they were an optional undergraduate course, containing the substance of Chapters 1-6, and part of Chapter 8. Once or twice they were given to graduate students in geometry, and then included also the bulk of Chapters 9-13. Chapter 7, with the part of Chapter 11 which depends on this, and the cubic transformations in Chapter 8, never figured in the course, but it seemed to me very desirable to add them to the published notes. There is of course much more that I would have liked to include (such as transformations at least of order 5, some study of the connexion between modular relations and the subgroups of finite index in the modular group, a general examination of rectification problems, and the parametrisation of confocal quadrics and of the tetrahedroid and wave surfaces); but a limit of length is laid down for this series of publications, which I fear I have already strained to the utmost.

In my treatment of elliptic functions I have tried above all to present a unified view of the subject as a whole, developing naturally out of the Weierstrass function; and to give the essential rudiments of every aspect of the subject, while unable to enter in very great detail into any one of these. In particular I have been concerned to emphasize the dependence of the properties of the functions on the shape of the lattice; it is for this reason that the modular function is introduced at such an early stage, and that equal prominence is given throughout (except in the context of the Jacobi functions) to the rhombic and the rectangular lattices.

The treatment of the theta functions will be seen to be rather slight. They are in themselves a large subject, of which our study is in a considerable measure independent, since our approach (based on Neville's) to the Jacobi functions obviates any need for the theta functions as a preliminary, except for the expression of invariants such as k, K, J in terms of τ or q, i. e. in terms of the lattice shape.

I have kept the analytic apparatus required to a minimum, largely because I am no expert analyst myself; all that I assume ought, I think, to be familiar to any graduate or third-year honours student, and is to be found in any such general textbook as Whittaker and Watson [43] or Copson [5]. For the study of elliptic curves I have of course had to assume some knowledge of algebraic geometry. The general theory sketched in Section 85 can be read up in detail in such works as van der Waerden [38] or Hodge and Pedoe [21]; and the properties of the genus used in Section 89 in any book on algebraic curves, such as Walker [40] or Semple and Kneebone [35]. For any assumed properties of the plane cubic and twisted quartic, probably the best sources are still the two classics of Salmon [32, 33], now available in modern reprints; and for the finite groups $\underline{V}$, $\underline{T}$, $\underline{O}$ etc. perhaps the easiest reference is my own monograph [10].

In conclusion, I would like to express my gratitude to the London Mathematical Society for making this publication possible; to the general editor of the series, Professor G. C. Shephard, for his patience; to Dr D. G. Larman for assistance with the bibliography; and particularly to my wife for her help in reading the proofs.

Istanbul, 1971 Patrick Du Val

Prof. Altamir A. R. Avaldi

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1 · Introductory

1.

For any complex number $z = x + iy$ (x, y real, $i^2 = -1$) we define $\mathrm{Re}(z) = x$, $\mathrm{Im}(z) = y$, $|z| = (x^2 + y^2)^{\frac{1}{2}}$, $\bar{z} = x - iy$. If $y = 0$ (i. e. if $\bar{z} = z$), z is real, if $y \neq 0$, z is imaginary, if $x = 0$, z is pure imaginary (note that 0 is pure imaginary without being imaginary) and if $|z| = 1$, z is unimodular. The real and pure imaginary axes in the Argand plane are horizontal and vertical respectively.

Lattices. A lattice Ω of complex numbers is an aggregate of complex numbers with the two properties: (i) Ω is a group with respect to addition; (ii) the absolute magnitudes of the non-zero elements are bounded below, i. e. there is a real number $k > 0$ such that $|\omega| \geq k$ for all $\omega \neq 0$ in Ω. Every lattice is either (i) trivial, consisting of 0 only; (ii) simple, consisting of all integer multiples of a single generating element, which is unique except for sign; or (iii) double, consisting of all linear combinations with integer coefficients of two generating elements ω_1, ω_2, whose ratio is imaginary. These are not unique; if ω_1, ω_2 generate Ω, so do

$$\omega_1' = p\omega_1 + q\omega_2, \qquad \omega_2' = r\omega_1 + s\omega_2,$$

where p, q, r, s are any integers satisfying $ps - qr = \pm 1$. It is usual however to require ω_1, ω_2 to be so ordered that $\mathrm{Im}(\omega_2/\omega_1)$ shall be positive; and if ω_1', ω_2' are to be similarly ordered, this requires $ps - qr = +1$.

2. Lattice shapes

If Ω is any lattice, and m any non zero complex number, $m\Omega$ denotes the aggregate of complex numbers $m\omega$ for all ω in Ω. This is also a lattice, which is said to be similar to Ω; similarity is an equivalence relation between lattices, an equivalence class being a lattice shape. All simple lattices are similar, i. e. constitute one lattice shape. The lattice points (i. e. elements of the lattice, represented as points in the Argand plane) are (for a simple lattice) at equal intervals along one line through the origin, but in general (for a double lattice) are the vertices of a pattern of parallelograms filling the whole plane, whose sides can be taken to be any pair of generators. The lattice point patterns for similar lattices are similar in the elementary sense.

$\overline{\Omega}$ denotes the aggregate of complex numbers $\overline{\omega}$ for all ω in Ω; $\overline{\Omega}$ is also a lattice. If $\overline{\Omega} = \Omega$, Ω is called real. This is the case if and only if either: (i) Ω is simple, its generator (and hence all its elements) being either real or pure imaginary; (ii) generators can be so chosen that ω_1 is real and ω_2 pure imaginary, in which case Ω is called rectangular, the lattice points being the vertices of a pattern of rectangles, whose sides are horizontal and vertical, i. e. parallel to the real and imaginary axes; or (iii) generators can be chosen which are conjugate complex, in which case Ω is called rhombic, the lattice points being the vertices of a pattern of rhombi, whose diagonals are horizontal and vertical. Any lattice similar to a rectangular or rhombic lattice is also rectangular or rhombic, but is only real if the sides of the rectangles (diagonals of the rhombi) are horizontal and vertical. The real rectangular or rhombic lattice will be called horizontal or vertical, according as the longer sides of the rectangles (longer diagonals of the rhombi) are horizontal or vertical.

Besides the simple lattice, there are two special lattice shapes: (i) square (ordinary squared paper pattern); this is both rectangular and rhombic, and may be said to be in the rectangular or rhombic position if the sides or diagonals respectively of the squares are horizontal and vertical (it is real in both cases); (ii) triangular (pattern of equilateral triangles filling the plane); this is rhombic in three ways, a rhombus (with diagonals in the ratio $\sqrt{3}:1$) consisting of any two triangles with a common side. Every lattice satisfies $\Omega = -\Omega$; the only cases in which $\Omega = k\Omega$, with $k \neq \pm 1$, are the square lattice $(\Omega = i\Omega)$ and the triangular lattice ($\Omega = \varepsilon\,\Omega$, where ε is a primitive cube root of unity; we shall throughout denote these cube roots by ε, ε^2 instead of the more usual ω, ω^2, to avoid confusion with the use of ω for an element of a lattice).

3. Residue classes

If z is any value of a complex variable, $z + \Omega$ denotes the aggregate of values $z + \omega$ for all ω in the lattice Ω. This aggregate is called a residue class $(\bmod\ \Omega)$. The residue classes $(\bmod\ \Omega)$ form a continuous group under addition, defined in the obvious way, namely $(z + \Omega) + (w + \Omega) = (z + w) + \Omega$. Ω itself is a residue class $(\bmod\ \Omega)$, the zero element of the group.

By a fundamental region of Ω we mean a simply connected region of the Argand plane which contains exactly one member of each residue class $(\bmod\ \Omega)$. If Ω is the trivial lattice, each residue class consists only of a single value of z, and the only fundamental region is the whole plane. If Ω is the simple lattice generated by ω, a fundamental region is an infinite strip, bounded by two parallel lines, one of which is the locus of $z + \omega$ for all z on the other; these bounding lines need not be perpendicular to ω, nor straight, though it is usually convenient to take them so; but they must not intersect. One of the two lines is included in the fundamental region, and the other is not, i. e. the strip is closed on one side and open on the other. If Ω is a double lattice, a fundamental region can be chosen in many ways; the simplest, and usually the most convenient, is what is called a unit cell, i. e. a parallelogram with sides ω_1, ω_2 (any pair of generators), including one of each pair of parallel sides, and one vertex, but excluding the rest of the boundary.

We obtain a topological model of the residue class group by identifying the points congruent $(\bmod\ \Omega)$ on the boundary of the fundamental region, i. e. joining up the open edges to the corresponding closed edges. For the simple lattice, identifying the points z, $z + \omega$ throughout the bounding lines of the strip, we obtain an infinite cylinder, with generators perpendicular to ω. This is topologically equivalent to a sphere with two pinholes, corresponding to the open ends of the cylinder (compare the Mercator map of the sphere, rolled up thus into a cylinder, on which every point of the sphere is mapped uniquely, except the two poles). For the double lattice, identifying points z, $z + \omega_2$ on the sides of the unit cell parallel to ω_1 we obtain a finite cylinder of length $|\omega_1|$ with ends perpendicular to its generators; to identify corresponding points on these two ends, the cylinder must be bent round (and also twisted, unless the sides of the unit cell are perpendicular, i. e. unless Ω is rectangular) to form a ring surface or torus.

In particular, the torus

$$x = (a + b\cos\phi)\cos\theta, \quad y = (a + b\cos\phi)\sin\theta, \quad z = b\sin\phi,$$
$$(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$$

$(a^2 > b^2 > 0)$, obtained by rotating the circle

$$(x - a)^2 + z^2 = b^2, \quad y = 0$$

about the z axis, is not only a topological model of the residue class group, but a conformal model of the fundamental region, for a rectangular lattice whose generators satisfy

$$\omega_2/\omega_1 = ib/\sqrt{(a^2 - b^2)}\ .$$

This means that the angle between the transverse common tangents of the two circles $(x \pm a)^2 + z^2 = b^2$, which are the section of the torus by the meridian plane $y = 0$, is equal to that between the diagonals of the rectangular unit cell of Ω.

Proof. The element of arc on the surface is given by

$$ds^2 = (a + b\cos\phi)^2 d\theta^2 + b^2 d\phi^2 = (a + b\cos\phi)^2(d\xi^2 + d\eta^2)\ ,$$

where $\xi = \theta$ and

$$\eta = \int \frac{b\,d\phi}{a + b\cos\phi} = \frac{2b}{\sqrt{(a^2 - b^2)}} \tan^{-1}\left(\sqrt{\left(\frac{a-b}{a+b}\right)} \tan \tfrac{1}{2}\phi\right)\ ,$$

so that the mapping of the point (θ, ϕ) of the torus on the point with cartesian coordinates (ξ, η) thus defined in a plane is conformal; and the torus, cut open along the meridian $\theta = \pm\pi$ and the parallel $\phi = \pm\pi$, is mapped (1, 1) on the rectangle between the lines

$$\xi = \pm\pi, \quad \eta = \pm b\pi/\surd(a^2 - b^2),$$

the fundamental region in question. //

No surface is known in three-dimensional Euclidean space, on which the residue class group modulo a non-rectangular lattice can be mapped in this way, so as to give at the same time a conformal map of the fundamental region. (Such a surface exists in eight-dimensional Euclidean space, but this is beyond our scope.)

4. Summation over a lattice

If Ω is any lattice and $f(z)$ any function of a complex variable, we shall denote by $\sum_{\Omega} f(\omega)$ the sum of $f(\omega)$ over all elements ω of Ω, and by $\sum_{\Omega}' f(\omega)$ the sum over all non-zero elements, i.e. the same sum with the term for $\omega = 0$ omitted.

Theorem 1.1. For any lattice Ω and any integer $n > 2$, $S_n(\Omega) = \sum_{\Omega}' \omega^{-n}$ converges absolutely.

Proof. It is well known that for $n > 1$, $\sum_{r=1}^{\infty} r^{-n}$ converges absolutely; denote this sum by s_n (it is in fact the Riemann zeta function $\zeta(n)$; but the use of the letter ζ here is unacceptable, since in the context of elliptic functions this letter has a quite different but equally well established meaning, to which we shall come later). If Ω is simple with generator ω, for even n, $S_n(\Omega) = 2\omega^{-n}s_n$, and for odd n, $S_n(\Omega) = 0$, as the terms $(r\omega)^{-n}$, $(-r\omega)^{-n}$ cancel. If Ω is a double lattice, the lattice points can be distributed into sets lying on the perimeters of a sequence of concentric parallelograms, similar to the unit cell, those on the r^{th} perimeter being of the form $p\omega_1 + q\omega_2$, where $|p|$, $|q|$ both $\leq r$, and at least one of them $= r$. Denote by $\sum_{r} f(\omega)$ the sum of terms with ω on the r^{th} perimeter; then $\sum_{\Omega}' f(\omega) = \sum_{r=1}^{\infty} \sum_{r} f(\omega)$. Now if h is the lesser diameter of the unit cell perpendicular to an edge, every ω on the r^{th} perimeter satisfies $|\omega| \geq rh$, the inequality being strict for most of them; and they are $8r$ in number. Thus $\sum_{r} |\omega|^{-n} < 8r(rh)^{-n}$, so that the series $\sum_{r=1}^{\infty} \sum_{r} |\omega|^{-n}$ is majorised by the absolutely convergent series $8h^{-n} \sum_{r=1}^{\infty} r^{1-n}$, and is thus itself absolutely convergent. //

The quantities $S_n(\Omega)$ thus defined clearly satisfy the homogeneity property $S_n(k\Omega) = k^{-n} S_n(\Omega)$, for all complex numbers $k \neq 0$ and all integers $n > 2$, since every term in the series on the left is k^{-n} times the corresponding term in that on the right. It follows that if n is odd, $S_n(\Omega) = 0$, for every lattice Ω, since $\Omega = -\Omega$, $S_n(\Omega) = S_n(-\Omega) = -S_n(\Omega)$. Similarly, if Ω is square, as $\Omega = i\Omega$, $S_n(\Omega) = 0$ for all n not divisible by 4, and if Ω is triangular, as $\Omega = \varepsilon\,\Omega$, $S_n(\Omega) = 0$ for all n not divisible by 6. If Ω is real, $S_n(\Omega)$ is real for all n, conjugate complex elements of the lattice giving rise to conjugate complex terms in the sum, and real elements to real terms; and in general $S_n(\overline{\Omega}) = \overline{S_n(\Omega)}$.

The simple lattice generated by ω can be regarded as the limit of a double lattice, of which one generator $\omega_1 = \omega$ remains constant, and the other ω_2 varies continuously in such a way that $\mathrm{Im}(\omega_2/\omega_1)$ tends to infinity, as all the lattice points except the integer multiples of ω recede to infinity, leaving the plane empty of lattice points except those of the simple lattice. The simple lattice will therefore be called a degenerate double lattice.

Theorem 1. 2. <u>When a double lattice Ω, varying continuously, tends to the degenerate limit, with generator ω, $S_n(\Omega)$ tends, uniformly in $\mathrm{Re}(\omega_2/\omega_1)$, to the limit $2\omega^{-n}s_n$, its value for the simple lattice.</u>

Proof. Denote ω_2/ω_1 by τ; on account of the homogeneity, it is sufficient to prove the theorem for the lattice Ω_τ, generated by 1, τ. Now for any even n, pairing off the equal terms for ω, $-\omega$, we can write

$$S_n(\Omega_\tau) = 2s_n + 2 \sum_{q=1}^{\infty} \sum_{p=-\infty}^{\infty} (p + q\tau)^{-n} .$$

Now let k be any integer. For each value of q, we can divide the values of p into sets of kq consecutive integers, according as $\mathrm{Re}(p + q\tau)$ lies between consecutive multiples of kq. If $\mathrm{Im}(\tau) > k$, whatever $\mathrm{Re}(\tau)$ may be, for the two such sets of values of p defined by

$$rkq \le \mathrm{Re}(p + q\tau)kq, < (r + 1)kq, \quad -(r + 1)kq \le \mathrm{Re}(p + q\tau) < -rkq$$

we have $|p + q\tau| > kq\sqrt{(1 + r^2)}$, so that for each value of q,

$$\left| \sum_{p=-\infty}^{\infty} (p + q\tau)^{-n} \right| < 2(kq)^{1-n} \sum_{r=0}^{\infty} (1 + r^2)^{-\frac{1}{2}n}$$

$$< 2(kq)^{1-n}(1 + 2^{-\frac{1}{2}n} + s_n) ,$$

replacing $(1 + r^2)^{-\frac{1}{2}n}$ by $(r - 1)^{-n}$, in all but the first two terms, since $(r - 1)^2 < 1 + r^2$. Hence

$$\left| 2 \sum_{q=1}^{\infty} \sum_{p=-\infty}^{\infty} (p + q\tau)^{-n} \right| < 4k^{1-n}s_{n-1}(1 + 2^{-\frac{1}{2}n} + s_n) ,$$

irrespective of the value of $\mathrm{Re}(\tau)$. Thus by taking $\mathrm{Im}(\tau)$ greater than a sufficiently large integer k, we can make

$$|S_n(\Omega_\tau) - 2s_n|$$

as small as we like, uniformly in $\mathrm{Re}(\tau)$; the theorem is thus proved for Ω_τ, and follows immediately for any $\Omega = \omega_1\Omega_\tau$. //

5. Functions and periods

We recall that a function $f(u)$ of a complex variable u is analytic at $u = a$ if it has an expansion as a power series $f(u) = \sum_{r=0}^{\infty} c_r(u - a)^r$, with constant coefficients $c_0, c_1, \ldots$, converging absolutely and uniformly in some circle $|u - a| < k$, where $k > 0$. $f(u)$ is meromorphic at $u = a$ if for some integer n, $(u - a)^n f(u)$ is analytic at $u = a$; if $n > 0$ is the least integer for which this holds, $f(u)$ has an expansion

$$f(u) = \sum_{r=1}^{n} b_r(u - a)^{-r} + \sum_{r=0}^{\infty} c_r(u - a)^r ,$$

with $b_n \neq 0$; in this case $u = a$ is a pole of $f(u)$, of order n; the terms $\sum_{r=1}^{n} b_r(u - a)^{-r}$ are called the infinite part of the function $f(u)$, b_n its leading coefficient, and b_1 its residue, at $u = a$. (This well established use of the word residue has of course nothing to do with residue classes, to which unfortunately we occasionally have to refer in the same contexts.) Similarly $u = a$ is a zero of order n of $f(u)$ if $f(u)$ is analytic at $u=a$, $f(a) = 0$, and n is the greatest integer such that $(u - a)^{-n}f(u)$ is analytic at $u = a$, i. e. c_n is the first non-zero coefficient in the expansion of $f(u)$ at $u = a$, which is accordingly of the form $f(u) = \sum_{r=n}^{\infty} c_r(u - a)^r$.

A function is said to be analytic or meromorphic in a given region, or in the whole plane, if it is so at every point of the region or of the plane. If $f(u)$ is analytic and non-zero at any point, in any region, or in the whole plane, so is $\frac{1}{f(u)}$; if $f(u)$ is meromorphic, so is $\frac{1}{f(u)}$, the poles of each being the zeros of the other, and of the same order. The poles of a function meromorphic in any region are a discrete set, i. e. for each pole, the distances of other poles from it are bounded below; and if $f(u)$ is meromorphic in any finite region, including its boundary, $f(u)$ can only have a finite number of poles in the region. As $\frac{1}{f(u)}$ is also meromorphic, $f(u)$ can only have a finite number of zeros in the region; and as $f(u) - c$ is meromorphic (for any constant c) $f(u)$ can only assume a given value c in a finite set of points in the region.

A period ω of a function $f(u)$ is a constant such that $f(u + \omega) = f(u)$ for all u. The sum of two periods is also trivially a period, and if ω is a period, so is $-\omega$. Thus the periods of any function form a group with respect to addition. On the other hand, unless the absolute magnitude of non-zero periods is bounded below, the function must be constant in any region in which it is differentiable, since $\frac{f(u+h) - f(u)}{h} = 0$ for some arbitrarily small but non-zero values of h. Thus the periods of a non-constant meromorphic function must be a lattice. Zero is of course a period of every function; if it is the only one, the lattice of periods is the trivial lattice, and the function is called non-periodic. If a function has a simple or double lattice of periods, it is called simply or doubly periodic. Familiar examples of simply periodic functions are $\sin u$, $\tan u$, e^u, with simple lattices of periods generated by 2π, π, $2i\pi$ respectively.

6. Definition

An elliptic function is a function of a complex variable, which is meromorphic in the whole plane, and doubly periodic. Since it has the same value in all points of any residue class (mod Ω), where Ω is its lattice of periods, it can be thought of as a function of the residue class, rather than of the individual value of u, i. e. a function of position on the torus model of the residue class group rather than of position in the plane. Before proving (by construction) the existence of some functions with these properties, it is convenient to prove some elementary consequences of the definition, assuming that such functions exist.

7. Liouville's theorem

This states that any function which is analytic and bounded in the whole plane is a constant. Also, a function which is analytic in any finite region (including its boundary) is bounded in that region. Hence, an elliptic function which has no residue classes of poles is bounded in the fundamental region, and so in the whole plane, and is accordingly a constant. This principle is applied in two main ways to elliptic functions:

Theorem 1. 3. If two elliptic functions have the same lattice of periods, the same residue classes of poles, and the same residue classes of zeros, of the same order in each case, the ratio of the two functions is a non-zero constant.

Proof. If $f(u)$, $g(u)$ have either zeros or poles of the same order at $u = a$, $\frac{f(u)}{g(u)}$ is analytic and non-zero at $u = a$. //

Theorem 1. 4. If two elliptic functions have the same lattice of periods, and the same residue classes of poles, with the same infinite part in each pole, the functions differ by a constant.

Proof. If $f(u)$, $g(u)$ have the same infinite part at $u = a$, $f(u) - g(u)$ is analytic there, i. e. has no pole. //

8. Contour integration theorems

For any function meromorphic in a simply connected region R bounded by a closed contour C, we recall the three classical theorems on integration round the contour C: Let $f(u)$ be meromorphic in R, with zeros of orders $m_1, \ldots, m_h$ at $u = a_1, \ldots, a_h$ and poles of order $n_1, \ldots, n_k$ at $u = b_1, \ldots, b_k$, with residues $r_1, \ldots, r_k$ respectively, all these zeros and poles being in R but none on C. Then

I. $\int_C f(u)du = 2\pi i \sum_{j=1}^{k} r_j$.

II. $\int_C \frac{f'(u)du}{f(u)} = 2\pi i(\sum_{j=1}^{h} m_j - \sum_{j=1}^{k} n_j)$.

III. $\int_C \frac{uf'(u)du}{f(u)} = 2\pi i(\sum_{j=1}^{h} m_j a_j - \sum_{j=1}^{k} n_j b_j)$.

From this we deduce

Theorem 1. 5. Let $f(u)$ be an elliptic function with the lattice Ω of periods, zeros of order $m_1, \ldots, m_h$ in the residue classes $a_1+\Omega, \ldots, a_h+\Omega$, and poles of order $n_1, \ldots, n_k$ in the residue classes $b_1+\Omega, \ldots, b_k+\Omega$, with residues $r_1, \ldots, r_k$ respectively. Then

I. $\sum_{j=1}^{k} r_j = 0$;

II. $\sum_{j=1}^{h} m_j = \sum_{j=1}^{k} n_j$;

III. $\sum_{j=1}^{h} m_j a_j \equiv \sum_{j=1}^{k} n_j b_j \pmod{\Omega}$.

Proof. Take the contour C to be the boundary of a unit cell, starting from a chosen point $u = c$, and travelling along straight lines to $u = c + \omega_1$, $c + \omega_1 + \omega_2$, $c + \omega_2$, and back to $u = c$ in turn, c being chosen so that the path does not pass through any zero or pole. If $\phi(u)$ is any function of u

$$\int_C \phi(u)du = \int_c^{c+\omega_1}(\phi(u) - \phi(u+\omega_2))du + \int_c^{c+\omega_2}(\phi(u+\omega_1) - \phi(u))du . \tag{8.1}$$

If $f(u)$ is an elliptic function with period lattice Ω, generated by ω_1, ω_2, so is $\frac{f'(u)}{f(u)}$, and in both the integrals I, II, the integrand in both terms on the right in (8.1) is identically zero, which gives the results I, II of the theorem. As for integral III, $\frac{uf'(u)}{f(u)}$ is not of course an elliptic function; but as in this case $\phi(u) - \phi(u + \omega_2) = -\omega_2 f'(u)/f(u)$, the first term on the right in (8.1) becomes

$$-\omega_2 \int_c^{c+\omega_1} d \log f(u) = \omega_2(\log f(c) - \log f(c + \omega_1)) ;$$

and as $f(c + \omega_1) = f(c)$, the difference between their logarithms as obtained from the integral must be an integer multiple of $2\pi i$, say $-2q\pi i$; thus the first term in (8.1) reduces to $2\pi i. q\omega_2$; and similarly the other term reduces to $2\pi i. p\omega_1$. Thus the integral III is equal to $2\pi i$ times an element $p\omega_1 + q\omega_2$ of Ω, which proves the result III of the theorem. //

9. Order of an elliptic function

Just as an s-ple zero of a polynomial $f(x)$ is commonly and conveniently regarded as being s coincident zeros of $f(x)$, or roots of the equation $f(x) = 0$, and this convention enables us to say that an equation of degree n has exactly n roots, when we make due allowance for coincidences; so an s-ple zero or pole of any meromorphic function is conventionally to be regarded as s coincident zeros or poles; and in the case of elliptic functions, with period lattice Ω, if $u = a$ is an s-ple zero or pole, so is every member of the residue class $a + \Omega$, which is regarded as s coincident residue classes of zeros or poles. With this convention the results II, III of Theorem 1.5 can be restated as

Theorem 1.6. An elliptic function $f(u)$ assumes any value c in a number n of residue classes which is independent of c and characteristic of $f(u)$, making due allowance for coincidences among these n residue classes for some values of c; moreover, the sum of these n residue classes is independent of c.

Proof. Results II, III of Theorem 1.5 mean that, making due allowance for coincidences, the number of residue classes of zeros is equal to that of residue classes of poles, and that the sum of the former residue classes is equal to that of the latter. Moreover, the residue classes in which $f(u) = c$ are those which are zeros of the function $f(u) - c$, which is an elliptic function with the same lattice of periods and the same poles as $f(u)$; thus the number and sum of the residue classes in which $f(u) = c$ are equal to the number and sum of the poles. //

The integer n of Theorem 1.6 is called the order of $f(u)$. We may compare the order of a rational function $\frac{f(x)}{g(x)}$, where $f(x)$, $g(x)$ are polynomials; the order is the greater of the degrees of $f(x)$, $g(x)$, and is the number of solutions of the equation $\frac{f(x)}{g(x)} = c$, for any c, when we make due allowance for coincident roots (and for infinite roots) of the polynomial equation $f(x) - cg(x) = 0$. Rather similarly, most important simply periodic functions have a definite order in this sense; $\sin x$, for instance, assumes any given value c in precisely two residue classes (mod 2π), which coincide for $c = 1$ and for $c = -1$.

No elliptic function can have order 1.

Proof. Any function of order 1 would have just one residue class of simple poles, whose residue cannot be zero, since the infinite part of the function at a simple pole $u = a$ must consist of a single term $b(u - a)^{-1}$, $b \neq 0$; and this contradicts result I of Theorem 1.5. Even more simply however, if $f(u)$ were an elliptic function of order 1, the relation $z = f(u)$ would define a one-one continuous mapping of the torus, model of the residue class group, onto the z sphere, which is manifestly impossible topologically. //

An elliptic function of order 2 must have either one residue class of double poles, with residue 0 (i. e. at which the infinite part consists of a single term in $(u - a)^{-2}$, and none in $(u - a)^{-1}$), or two residue classes of simple poles with equal and opposite residues.

For any elliptic function $f(u)$ of order n, there can only be a finite number of values c for which the n residue classes in which $f(u) = c$ are not all distinct. For apart from any multiple poles $(c = \infty)$, the places in which $f(u) = c$ has a multiple root for any finite c are the stationary points of f(u), i. e. the zeros of f'(u), which is also an elliptic function, and vanishes in only a finite number of residue classes. // In fact

Theorem 1. 7. Let f(u) be an elliptic function of order n, and the sum of the n residue classes in which f(u) assumes any given value be $w + \Omega$; and let $f(u) = c_i$ in s_i coincident residue classes $a_i + \Omega$ $(i = 1, 2, \ldots)$. Then

$$\Sigma(s_i - 1) = 2n, \quad \Sigma(s_i - 1)a_i \equiv 2w \pmod{\Omega},$$

the summation being over all residue classes $a_i + \Omega$ for which $s_i > 1$.

Proof. The result is clearly unaltered if we include in the summation any further residue classes $a_i + \Omega$ for which $s_i = 1$. We therefore specifically include all the poles of f(u), whether simple or multiple; let the poles $(c_i = \infty)$ be given by $i = 1, \ldots, h$, and other (finite) multiple values c_i by $i = h+1, \ldots, k$; then

$$\sum_{i=1}^{h} s_i = n, \qquad \sum_{i=1}^{h} s_i a_i \equiv w \pmod{\Omega}.$$

As an s_i-ple pole of any meromorphic function is an (s_i+1)-ple pole of its derivative, f'(u) is of order $\sum_{i=1}^{h}(s_i + 1)$, and the sum of its residue classes of poles is $\sum_{i=1}^{h}(s_i + 1)a_i + \Omega$. The remaining residue classes $a_i + \Omega$ $(i = h+1, \ldots, k)$ are the zeros of f'(u), a_i being an s_i-ple zero of $f(u) - c_i$, and hence an (s_i-1)-ple zero of f'(u). Thus

$$\sum_{i=h+1}^{k}(s_i - 1) = \sum_{i=1}^{h}(s_i + 1) = 2n - \sum_{i=1}^{h}(s_i - 1),$$

$$\sum_{i=h+1}^{k}(s_i - 1)a_i \equiv \sum_{i=1}^{h}(s_i + 1) \equiv 2w - \sum_{i=1}^{h}(s_i - 1)a_i \pmod{\Omega},$$

i. e. $$\sum_{i=1}^{k}(s_i - 1) = 2n, \qquad \sum_{i=1}^{k}(s_i - 1)a_i \equiv 2w \pmod{\Omega},$$

which proves the theorem. //

In particular, if n = 2, there are four values of c (necessarily all distinct, and possibly including infinity) for which the two residue classes in which $f(u) = c$ coincide. If $a + \Omega$, $b + \Omega$ are two of these, $2a \equiv 2b \pmod{\Omega}$; thus the four are

$$a + \Omega, \quad a + \tfrac{1}{2}\omega_1 + \Omega, \quad a + \tfrac{1}{2}\omega_2 + \Omega, \quad a + \tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2 + \Omega,$$

for some a, and the union of all four is a residue class $(\text{mod } \tfrac{1}{2}\Omega)$.

2 · The Weierstrass functions

10. The Weierstrass function $\wp u$

Largely for historical reasons, it is usual to denote a pair of generators of the period lattice of an elliptic function by $2\omega_1$, $2\omega_2$; we shall therefore denote the lattice itself by 2Ω, where Ω is generated by ω_1, ω_2. When it is desirable to specify the period lattice with which a given elliptic function $f(u)$ has been constructed we shall write it $f(u|2\Omega)$, but where only one lattice is under consideration we shall write simply $f(u)$, or even in some cases fu.

In the first place, we define for any integer $n \geq 3$

$$P_n(u|2\Omega) = \sum_{\Omega} (u - 2\omega)^{-n} .$$

This series converges absolutely and uniformly in every finite region which finitely excludes all lattice points, such as $|u| < K$, $|u-2\omega| > k$, where K, k are positive constants which can be taken as large and as small as we like.

Proof. For all u in the region, and for all ω such that $|\omega| > K$ (i. e. for all but a finite number of terms in the series) $|u-2\omega| > |\omega|$, so that the series is majorized by that for $S_n(\Omega)$. // By taking K large enough and k small enough, the region of convergence can be made to include any chosen point of the plane, except a lattice point $u = 2\omega$; thus $P_n(u)$ is analytic except at the lattice points. These are poles of order n, with infinite part $(u - 2\omega)^{-n}$, since by omitting this one term from the series, the point $u = 2\omega$ can be included in the region of convergence. If ω_0 is any element of Ω, $2\omega_0$ is a period of $P_n(u)$, since the substitution of $u + 2\omega_0$ for u merely permutes the terms of the series amongst themselves, as the aggregates Ω, $\Omega - \omega_0$ are the same; and (the convergence being absolute) does not affect the sum. Thus $P_n(u|2\Omega)$ is an elliptic function of order n, with period lattice 2Ω. Further, differentiating the series term by term, we have $P'_n(u) = -nP_{n+1}(u)$.

For $n = 2$ however, the corresponding series does not converge, as that which might be expected to define $S_2(2\Omega)$ does not converge. To obtain an elliptic function of order 2 in this sequence we define

$$\wp u = \wp(u|2\Omega) = u^{-2} + \sum_{\Omega}{}' \{(u - 2\omega)^{-2} - (2\omega)^{-2}\} .$$

This series converges absolutely and uniformly in the same type of region ($|u| < K$, $|u - 2\omega| > k$ for all ω in Ω) as that in which the series defining $P_n(u)$ ($n \geq 3$) converges.

Proof. As

$$(u - 2\omega)^{-2} - (2\omega)^{-2} = \tfrac{1}{4}u\omega^{-3}(1 - \tfrac{u}{4\omega})(1 - \tfrac{u}{2\omega})^{-2} ,$$

and if $|u| < K$ and $|\omega| > K$, then $|1 - \frac{u}{4\omega}| < \frac{5}{4}$ and $|1 - \frac{u}{2\omega}| > \frac{1}{2}$, we have for all u in the region, and all but a finite number of terms in the series

$$|(u - 2\omega)^{-2} - (2\omega)^{-2}| < \tfrac{5}{4}K\omega^{-3}$$

so that the series is majorised by that for $\frac{5}{4}KS_3(\Omega)$. // Thus $\wp u$ is analytic in the whole plane, except for the lattice points 2Ω, where it has poles of order 2, with infinite part $(u - 2\omega)^{-2}$. It is an even function, i. e. $\wp(-u) = \wp u$, since the substitution of $-u$ for u in the series merely interchanges each pair of terms given by elements $\pm\omega$ in Ω, and the one unpaired term u^{-2} is unchanged.

Differentiating the series term by term, $\wp' u = -2P_3(u)$. From the fact that $\wp' u$ is an elliptic function, it does not at once follow that $\wp u$ is one; integrating the identity

$$\wp'(u + 2\omega) = \wp' u$$

for any ω in Ω, we have only

$$\wp(u + 2\omega) = \wp u + c ,$$

for some constant c; but putting $u = -\omega$ in this,

$$\wp\, \omega = \wp(-\omega) + c = \wp\, \omega + c$$

since $\wp\, u$ is even; thus $c = 0$, and

$$\wp(u + 2\omega) = \wp\, u$$

for all ω in Ω. Thus $\wp\, u$ is an elliptic function of order 2 with period lattice 2Ω, and double poles in the zero residue class.

11. The differential equation satisfied by $\wp\, u$, and its expansion at the origin

The binomial expansion of $(u - 2\omega)^{-2} = \frac{1}{4}\omega^{-2}(1 - \frac{u}{2\omega})^{-2}$ converges absolutely for $|u| < 2|\omega|$; thus for $|u| < 2k$, where $k = \min|\omega|$ for all $\omega \neq 0$ in Ω, we can substitute this expansion (omitting the first term, cancelled by the term $-(2\omega)^{-2}$) in each term of the series for $\wp\, u$, and, the convergence still being absolute, collect the terms containing like powers of u, obtaining the series

$$\wp\, u = u^{-2} + \sum_{n=1}^{\infty} (2n + 1)S_{2n+2}(2\Omega)u^{2n}$$

as the expansion of $\wp\, u$ in its circle of convergence $|u| < 2k$, the odd powers of u being all absent (as we expect for an even function) since the coefficient of u^{2n-1} has the factor $S_{2n+1}(2\Omega) = 0$. Differentiating this term by term we have

$$\wp\,'u = -2u^{-3} + \sum_{n=1}^{\infty} 2n(2n + 1)S_{2n+2}(2\Omega)u^{2n-1}.$$

For brevity writing S_n for $S_n(2\Omega)$, the first few terms of these are

$$\wp\, u = u^{-2} + 3S_4u^2 + 5S_6u^4 + 7S_8u^6 + 9S_{10}u^8 + \dots,$$

$$\wp\,'u = -2u^{-3} + 6S_4u + 20S_6u^3 + 42S_8u^5 + 72S_{10}u^7 + \dots;$$

from which we have

$$\wp^3 u = u^{-6} + 9S_4 u^{-2} + 15S_6 + (27S_4^2 + 21S_8)u^2 + (120S_4S_6 + 27S_{10})u^4 + \ldots$$

$$\wp'^2 u = 4u^{-6} - 24S_4 u^{-2} - 80S_6 + (36S_4^2 - 168S_8)u^2 + (240S_4S_6 - 288S_{10})u^4 + \ldots$$

so that

$$\wp'^2 u - 4\wp^3 u + 60S_4 \wp u + 140S_6$$
$$= (108S_4^2 - 252S_8)u^2 + (660S_4S_6 - 396S_{10})u^4 + \ldots \qquad (11.1)$$

The left hand member of (11.1) is an elliptic function with period lattice 2Ω, and is analytic except possibly at the lattice points, the zero residue class, which are the only poles of the individual terms. The right hand member is the expansion of this same function in the neighbourhood $|u| < k$ of the origin, and shows that the function is analytic and equal to zero there. Since then the function has no pole at the origin, it has none at any of the other lattice points, and by Liouville's theorem is a constant, and equal to 0, its value at the origin. Thus identically in u,

$$\wp'^2 u = 4\wp^3 u - 60S_4(2\Omega)\wp u - 140S_6(2\Omega),$$

or defining $g_2 = g_2(2\Omega) = 60S_4(2\Omega)$, $g_3 = g_3(2\Omega) = 140S_6(2\Omega)$

$$\wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3; \qquad (11.2)$$

and $x = \wp u$ is a solution of the differential equation

$$\left(\frac{dx}{du}\right)^2 = 4x^3 - g_2 x - g_3. \qquad (11.3)$$

Note that all the coefficients in the series which is the right-hand member of (11.1) must be zero; thus $7S_8 = 3S_4^2$, $3S_{10} = 5S_4S_6$, ..., and each $S_{2n}(2\Omega)$ in turn can be expressed as a polynomial, with rational coefficients independent of Ω, in those for lower values of n, and hence by recursion as a polynomial in $S_4(2\Omega)$, $S_6(2\Omega)$, i.e. in g_2, g_3. In terms of these the expansion of $\wp u$ in its circle of convergence $|u| < k$ is

$$\wp u = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \frac{g_2^2}{1200}u^6 + \frac{g_2 g_3}{560}u^8 + \ldots, \qquad (11.4)$$

in which each coefficient is a polynomial in g_2, g_3, with rational coefficients, independent of Ω; thus if the constants g_2, g_3 are given, the whole expansion is uniquely determined.

Now as any point $u = a$, except the lattice points 2Ω, the poles of $\wp u$, can be reached from the origin by a continuous path in the u plane which is nowhere (except at the origin) at a distance $< d$ (for some $d > 0$) from any lattice point, the expansion of the function at $u = a$ is completely determined by analytic continuation from that at the origin, i. e. by the values of g_2, g_3. Hence if lattices Ω, Ω' satisfy $g_2(2\Omega') = g_2(2\Omega)$ and $g_3(2\Omega') = g_3(2\Omega)$, then also $\wp(u|2\Omega') = \wp(u|2\Omega)$ at every point at which either is analytic, and hence in fact everywhere, so that $\Omega' = \Omega$. Moreover, if $g_2^3(2\Omega')/g_3^2(2\Omega') = g_2^3(2\Omega)/g_3^2(2\Omega)$, then Ω', Ω are similar, for in this case there exists a constant $m \neq 0$ such that $g_2(2\Omega') = m^{-4}g_2(2\Omega) = g_2(2m\Omega)$, $g_3(2\Omega') = m^{-6}g_3(2\Omega) = g_3(2m\Omega)$, by the homogeneity of $S_4(2\Omega)$, $S_6(2\Omega)$, and hence $\Omega' = m\Omega$. We recall also that if Ω is square, $S_n(\Omega) = 0$ unless n is a multiple of 4, and if Ω is triangular, $S_n(\Omega) = 0$ unless n is a multiple of 6; thus $g_2 = 0$ for the triangular and $g_3 = 0$ for the square lattice. There cannot be a proper double lattice for which $g_2 = g_3 = 0$, since in this case the equation (11. 3) would reduce to $(\frac{dx}{du})^2 = 4x^3$, whose complete solution is $x=(u-c)^{-2}$, c being a constant of integration, and this is certainly not an elliptic function.

12. Stationary values of $\wp u$

As $\wp u$ has double poles in the zero residue class, the sum of the two residue classes in which it assumes any given value is zero. This follows also from the fact that it is even, $\wp(-u) = \wp u$. From this and the periodic property $\wp(u + 2\omega) = \wp u$, it follows that $\wp(2\omega - u) = \wp u$, i. e. $\wp u$ is an even function, not only of u, but of $u - \omega$, for all ω in Ω. If one residue class in which $\wp u = e$ is $\omega + 2\Omega$, where ω is in Ω but not in 2Ω, i. e. is a half period of $\wp u$, the other residue class for the same value coincides with it, i. e. every point of this residue class is a double zero of $\wp u - e$, and a stationary point of $\wp u$, hence also a zero of $\wp' u$. There are three such residue classes of half periods, namely $\omega_i + 2\Omega$ $(i = 1, 2, 3)$, where ω_1, ω_2 are any pair of generators of Ω, and $\omega_3 = -(\omega_1 + \omega_2)$; the union of these three residue classes with the null residue class is the lattice Ω. These are the three residue classes of zeros of $\wp'u$, which as we have seen is of order 3, its only poles being triple, in the zero residue class.

The stationary values $e_i = \wp\,\omega_i$ $(i = 1, 2, 3)$ of $\wp u$ are all distinct; for if $e_i = e_j = e$ say $(j \neq i)$, $\wp u - e$ would have two residue classes of double zeros, whereas it is of order 2 only, its poles being those of $\wp u$. The stationary values e_1, e_2, e_3 are the roots of the cubic equation $4x^3 - g_2x - g_3 = 0$, since from (11.2) $\wp'u = 0$ if and only if $\wp u$ is a root of this equation. Hence

$$e_1+e_2+e_3 = 0, \quad e_2e_3+e_3e_1+e_1e_2 = -\tfrac{1}{4}g_2, \quad e_1e_2e_3 = \tfrac{1}{4}g_3 . \qquad (12.1)$$

Moreover, as the discriminant of the cubic $4x^3 - g_2x - g_3$ is $\Delta = g_2^3 - 27g_3^2$, and we have just seen that e_1, e_2, e_3 are all distinct, $\Delta \neq 0$, i.e. $g_2^3 \neq 27g_3^2$, for any proper lattice 2Ω.

13. Homogeneity of $\wp u$, $\wp'u$

From the definition of $\wp u$, $P_n(u)$ as double series it is clear that for any $m \neq 0$,

$$\wp(mu\,|\,2m\Omega) = m^{-2}\wp(u\,|\,2\Omega), \quad \wp'(mu\,|\,2m\Omega) = m^{-3}\wp'(u\,|\,2\Omega), \qquad (13.1)$$

and in general

$$P_n(mu\,|\,2m\Omega) = m^{-n}P_n(u\,|\,2\Omega) ;$$

since on substituting simultaneously mu for u and $m\omega$ for every ω in Ω, each term in the series is multiplied by m^{-2}, m^{-n}. In particular, as the square lattice satisfies $i\Omega = \Omega$, for this lattice we have $\wp(iu) = -\wp u$, $\wp'(iu) = i\wp'u$. Similarly for the triangular lattice $\varepsilon\Omega = \Omega$, $\wp(\varepsilon u) = \varepsilon\wp u$, $\wp'(\varepsilon u) = \wp'u$.

14. Translation formula and quarter period values

Theorem 2.1

$$\wp(u \pm \omega_i) = \frac{(e_i-e_j)(e_i-e_k)}{\wp u - e_i} + e_i = \frac{e_i\wp u+(2e_i^2-\frac{1}{4}g_2)}{\wp u - e_i} . \qquad (14.1)$$

Proof. The function $\wp u - e_i$ has double poles in the zero residue class 2Ω and double zeros in $\omega_i + 2\Omega$; $\wp(u - \omega_i) - e_i$ on the other hand has double poles in $\omega_i + 2\Omega$, and double zeros in 2Ω; thus the product $(\wp u - e_i)(\wp(u - \omega_i) - e_i)$ has no poles, those of each factor being cancelled by the zeros of the other, and is accordingly a constant, by Liouville's theorem. The value of the constant product can be found by substituting either of the other half periods ω_j, ω_k for u, when one factor becomes $e_j - e_i$ and the other $e_k - e_i$. Thus

$$(\wp u - e_i)(\wp(u - \omega_i) - e_i) = d_i^2, \tag{14.2}$$

where we define

$$d_i^2 = (e_i - e_j)(e_i - e_k) = 3e_i^2 - \tfrac{1}{4}g_2 \tag{14.3}$$

by (12.1). Thus

$$\wp(u - \omega_i) = \frac{d_i^2}{\wp u - e_i} + e_i = \frac{e_i \wp u + (d_i^2 - e_i^2)}{\wp u - e_i}$$

which is equivalent to the theorem, by (14.3); obviously from the periodicity of $\wp u$, $\wp(u + \omega_i) = \wp(u - \omega_i)$. //

Further, if $-u \equiv u - \omega_i \pmod{2\Omega}$, i.e. if $u \equiv \tfrac{1}{2}\omega_i \pmod{\Omega}$, we have $(\wp u - e_i)^2 = d_i^2$, i.e. $\wp u = e_i \pm d_i$. There are four residue classes $(\mathrm{mod}\ 2\Omega)$ satisfying this condition, namely $\pm \tfrac{1}{2}\omega_i + 2\Omega$, $\pm(\tfrac{1}{2}\omega_i \pm \omega_j) + 2\Omega$, where $j \neq i$; it does not matter which of the two suffixes other than i is used here, as $\tfrac{1}{2}\omega_i \pm \omega_j \equiv -\tfrac{1}{2}\omega_i \pm \omega_k \pmod{2\Omega}$. We define the square root d_i so that

$$\wp(\tfrac{1}{2}\omega_i) = e_i + d_i, \quad \wp(\tfrac{1}{2}\omega_i + \omega_j) = e_i - d_i \tag{14.4}$$

15. Addition and duplication formulae

Theorem 2.2.

$$\wp(v + w) = \frac{1}{4}\left(\frac{\wp' v - \wp' w}{\wp v - \wp w}\right)^2 - \wp v - \wp w \tag{15.1}$$

$$\wp(2v) = \frac{1}{4}\frac{\wp''^2 v}{\wp'^2 v} - 2\wp v \qquad (15.2)$$

(Of these two, (15. 1) is known as the addition theorem or formula, (15. 2) as the duplication formula. (15. 1) is an algebraic addition formula, like for instance those for the trigonometric functions, in that it expresses $\wp(v+w)$ as an algebraic, though not of course a rational, function of $\wp v$, $\wp w$ only.)

Proof. For any constants A, B, $f(u) = \wp' u - A\wp u - B$ is an elliptic function of order 3 with triple poles in the zero residue class. The constants A, B can be so chosen that f(u) shall vanish in two assigned residue classes $v + 2\Omega$, $w + 2\Omega$, or so that f(u) shall have double zeros in an assigned residue class $v + 2\Omega$:

$$\left.\begin{array}{l}\wp' v - A\wp v - B = 0\\ \wp' w - A\wp w - B = 0\end{array}\right\} \Longrightarrow A = \frac{\wp' v - \wp' w}{\wp v - \wp w}, \quad B = \frac{\wp v\,\wp' w - \wp w\,\wp' v}{\wp v - \wp w} \qquad (15.3)$$

$$\left.\begin{array}{l}\wp' v - A\wp v - B = 0\\ \wp'' v - A\wp' v = 0\end{array}\right\} \Longrightarrow A = \frac{\wp'' v}{\wp' v}, \quad B = \frac{\wp'^2 v - \wp v\,\wp'' v}{\wp' v}. \qquad (15.4)$$

Since $\wp' u = f(u) + A\wp u + B$, f(u) and $\wp u$ satisfy the identity

$$(f(u) + A\wp u + B)^2 = 4\wp^3 u - g_2\wp u - g_3,$$

which on putting $f(u) = 0$ gives

$$4\wp^3 u - A^2\wp^2 u - (2AB + g_2)\wp u - (B^2 + g_3) = 0.$$

Thus the roots of the equation

$$4x^3 - A^2x^2 - (2AB + g_2)x - (B^2 + g_3) = 0$$

are the values of $\wp u$ in the three residue classes of zeros of f(u), namely $\wp v$, $\wp w$, $\wp(v+w)$ if $f(v) = f(w) = 0$, or $\wp v$, $\wp v$, $\wp(2v)$ if $f(v) = f'(v) = 0$. Hence, as the sum of the three roots is $\frac{1}{4}A^2$, taking the value of A from (15. 3), (15. 4) in the two cases, we obtain (15. 1), (15. 2) respectively, and the theorem is proved. //

Writing for brevity $\wp(v+w) = x$, $\wp v = y$, $\wp w = z$ in (15. 1), substituting $(4y^3 - g_2 y - g_3)^{\frac{1}{2}}$, $(4z^3 - g_2 z - g_3)^{\frac{1}{2}}$ for $\wp' v$, $\wp' w$, rationalising and clearing of fractions, we have after trivial simplification

$$y^2z^2+z^2x^2+x^2y^2-2xy(x+y+z)+\tfrac{1}{2}g_2(yz+zx+xy)+g_3(x+y+z)+\tfrac{1}{16}g_2^2=0. \quad (15.5)$$

This is satisfied by $x = \wp u$, $y = \wp v$, $z = \wp w$ if and only if $u \pm v \pm w = 0$. Taking it as a quadratic equation for x whose roots are $\wp(v + w)$, $\wp(v - w)$, we have (still writing y, z for $\wp v$, $\wp w$)

$$\wp(v+w) + \wp(v-w) = \frac{(y+z)(2yz-\frac{1}{2}g_2)-g_3}{(y-z)^2} \quad (15.6)$$

$$\wp(v+w).\ \wp(v-w) = \frac{y^2z^2+\frac{1}{2}g_2 yz+g_3(y+z)+\frac{1}{16}g_2^2}{(y-z)^2}\ . \quad (15.7)$$

Another form of the condition that $u + v + w = 0$ is

$$\begin{vmatrix} \wp u & \wp' u & 1 \\ \wp v & \wp' v & 1 \\ \wp w & \wp' w & 1 \end{vmatrix} = 0 \quad (15.8)$$

since if and only if $u + v + w = 0$, $\wp' u - A \wp u - B = \wp' v - A \wp v - B = \wp' w - A \wp w - B = 0$ for some constants A, B.

The duplication formula (15.2) expresses $\wp(2v)$ not only as an algebraic but as a rational function of $\wp v$; differentiating (11.2) on both sides we have $\wp'' u = 6 \wp^2 u - \frac{1}{2}g_2$; thus substituting $4y^3-g_2y-g_3$, $6y^2-\frac{1}{2}g_2$ for $\wp'^2 v$, $\wp'' v$ (still writing y for $\wp v$),

$$\wp(2v) = \frac{y^4 + \frac{1}{2}g_2 y^2 + 2g_3 y + \frac{1}{16}g_2^2}{4y^3 - g_2 y - g_3} \quad (15.9)$$

By repeated application of the addition and duplication formulae, expressions for $\wp(nu)$ for all positive integers n can be found recursively, as rational functions of $\wp u$.

Since $2u \equiv -u \pmod{2\Omega}$ if and only if u is in the zero residue class, or in one or other of the four pairs of residue classes

$$\pm\tfrac{2}{3}\omega_1,\quad \pm\tfrac{2}{3}\omega_2,\quad \pm\tfrac{2}{3}\omega_3,\quad \pm\tfrac{2}{3}\omega_4$$

(where $\omega_4 = \omega_1 - \omega_2$), the values of $\wp(\frac{2}{3}\omega_i)$ $(i = 1, 2, 3, 4)$ are the roots of the quartic equation

$$x^4 - \tfrac{1}{2}g_2x^2 - g_3x - \tfrac{1}{48}g_2^2 = 0\,, \qquad (15.10)$$

obtained by writing x for y, equating $\wp(2v)$ also to x, and clearing of fractions, from (15.9). Similarly for any integer n, by repetition of the addition and duplication formulae, we can obtain an algebraic equation whose roots are the values of $\wp u$ for all u such that nu is in 2Ω, but ru is not for any positive integer $r < n$; such values of u we may call primitive n^{th} periods; they clearly form $n^2 - 1$ residue classes if n is prime, and fewer than this if n is not prime.

16. Functions of order 2

Theorem 2.3. Every elliptic function of order 2 can be expressed in the form

$$f(u) = \frac{a\,\wp(u - u_0) + b}{c\,\wp(u - u_0) + d}\,, \qquad (16.1)$$

where u_0, a, b, c, d are constants, and $ad - bc \neq 0$.

Proof. If f(u) has a double pole, let this be in the residue class $u_0 + 2\Omega$; then f(u) vanishes in a pair of residue classes $u_0 \pm k + 2\Omega$, for some k (coincident if k is in Ω). $\wp(u - u_0) - \wp k$ is an elliptic function with the same zeros and the same poles as f(u), and of the same multiplicities, so that their quotient is an elliptic function without poles, and hence a constant A, i.e. $f(u) = A(\wp(u-u_0) - \wp k)$, which is of the form (16.1) with $c = 0$. On the other hand, if f(u) has two residue classes of simple poles, let these be $v + 2\Omega$, $w + 2\Omega$; they can be taken to be $u_0 \pm h + 2\Omega$, where $2u_0 \equiv v + w$, $2h \equiv v - w \pmod{2\Omega}$, which determines $u_0 + 2\Omega$ as any one of four residue classes (mod 2Ω) whose union is a residue class (mod Ω), the difference of any two being a half period ω_i (mod 2Ω). The zeros are again of the form $u_0 \pm k + 2\Omega$, for some k; thus the function

$$\frac{\wp(u - u_0) - \wp k}{\wp(u - u_0) - \wp h}$$

k and complementary modulus k' by

$$k^2 = \frac{e_3 - e_2}{e_1 - e_2}, \quad k'^2 = \frac{e_1 - e_3}{e_1 - e_2}, \quad \text{so that } k^2 + k'^2 = 1, \tag{16.2}$$

but of course which these are, out of the twelve square roots of the six cross-ratios, depends on the choice of generators ω_1, ω_2 for Ω, and on an arbitrary choice of each square root. The twelve possible values of the modulus for a given lattice are

$$\pm k, \quad \pm k', \quad \pm\frac{1}{k}, \quad \pm\frac{1}{k'}, \quad \pm\frac{ik}{k'}, \quad \pm\frac{ik'}{k} \; ; \tag{16.3}$$

and it is sometimes convenient to define the modular angle θ, such that $k = \sin\theta$, $k' = \cos\theta$, so that the same twelve values are

$$\pm\sin\theta, \; \pm\cos\theta, \; \pm\csc\theta, \; \pm\sec\theta, \; \pm i\tan\theta, \; \pm i\cot\theta \, .$$

We note also that the moduli are the same for all lattices similar to a given one, i. e. depend only on the lattice shape, since replacing Ω by $m\Omega$ merely replaces e_1, e_2, e_3 by $m^{-2}e_1$, $m^{-2}e_2$, $m^{-2}e_3$.

17. The elliptic function field for a given lattice

Since the sum, difference, and product of any two elliptic functions with a given period lattice 2Ω, and the reciprocal of any such function except the zero constant, is again an elliptic function with the period lattice 2Ω, and since of course the addition and multiplication of these functions is subject to all the usual laws (commutative, associative, and distributive), the aggregate of elliptic functions with the period lattice 2Ω constitutes a field, which we shall denote by $\mathsf{E}(2\Omega)$. (In the same way of course, the aggregate of functions meromorphic in the whole plane constitutes a field, of which $\mathsf{E}(2\Omega)$ is a subfield, for every lattice 2Ω.) $\mathsf{E}(2\Omega)$ contains the field C of complex numbers, the constant functions, i. e. it is an extension of C. The algebraic properties of this extension are contained in the following theorems:

Theorem 2. 5. Every even elliptic function is identically expressible as a rational function (with complex coefficients) of the function

$\wp u$ with the same period lattice.

Proof. Let the function f(u), satisfying f(-u) = f(u), have zeros of order m_i in the residue classes $\pm a_i + 2\Omega$ (i = 1, ..., h), and poles of order n_i in the residue classes $\pm b_i + 2\Omega$ (i = 1, ..., k), apart from any zero or pole there may be in the null residue class 2Ω. If any of these residue classes is a half period, i. e. a_i or $b_i = \omega$ say is in Ω but not in 2Ω, its order must be even, since f(u) is an even function not only of u but of $u - \omega$; and we count zeros or poles of order 2m in this residue class as two superimposed sets of order m, one in $\omega + 2\Omega$, and the other in $-\omega + 2\Omega$, which is the same residue class. The zero residue class 2Ω consists of 2s-ple poles, (-2s)-ple zeros, or neither, according as the difference

$$s = \sum_{i=1}^{h} m_i - \sum_{i=1}^{k} n_i$$

is positive, negative, or zero. Now the function

$$F(u) = \prod_{i=1}^{h} (\wp u - \wp a_i)^{m_i} / \prod_{i=1}^{k} (\wp u - \wp b_i)^{n_i}$$

has clearly the same zeros and poles, and of the same order, as f(u); hence by Liouville's theorem $\frac{f(u)}{F(u)}$ is a constant K say, i. e.

$$f(u) = K \prod_{i=1}^{h} (\wp u - \wp a_i)^{m_i} / \prod_{i=1}^{k} (\wp u - \wp b_i)^{n_i},$$

which is the required expression. //

Theorem 2. 6. Every elliptic function with the given period lattice 2Ω can be expressed as $P(\wp u) + \wp' u Q(\wp u)$, where P(x), Q(x) are rational functions with complex coefficients.

Proof. If f(u) is any function of u, f(u) + f(-u) is an even function, and f(u) - f(-u) is an odd one; and as $\wp' u$ is also an odd function, $\frac{(f(u) - f(-u))}{\wp' u}$ is an even one. Thus by Theorem 2. 5, if f(u) is any elliptic function with the period lattice 2Ω,

$$\tfrac{1}{2}(f(u) + f(-u)) = P(\wp u), \quad \frac{\tfrac{1}{2}(f(u) - f(-u))}{\wp' u} = Q(\wp u), \tag{17.1}$$

where $P(x)$, $Q(x)$ are rational functions over $\mathbf{C}$; and from (17.1) the expression $f(u) = P(\wp u) + \wp' u Q(\wp u)$ follows. //

Theorem 2.7. Every elliptic function $f(u)$ with period lattice 2Ω satisfies identically a quadratic equation, whose coefficients are rational functions of $\wp u$ over $\mathbf{C}$.

Proof. Let $f(u) = P(\wp u) + \wp' u Q(\wp u)$, then $\wp' u = (f(u) - P(\wp u))/Q(\wp u)$, and

$$(f(u) - P(\wp u))^2 = Q^2(\wp u).(4\wp^3 u - g_2 \wp u - g_3),$$

which is the required equation. // Multiplying this by the square of the least common denominator of the rational functions $P(\wp u)$, $Q(\wp u)$, it becomes a polynomial equation in $f(u)$, $\wp u$, quadratic in the former, with coefficients in $\mathbf{C}$.

Theorem 2.8. Any two elliptic functions with the same period lattice 2Ω satisfy identically an algebraic equation with coefficients in $\mathbf{C}$.

Proof. $f(u)$, $g(u)$ satisfy identically

$$F(f(u), \wp u) = 0, \quad G(g(u), \wp u) = 0,$$

where $F(x, z)$, $G(y, z)$ are polynomials with coefficients in $\mathbf{C}$; and hence also $H(f(u), g(u)) = 0$, where $H(x, y)$ is the resultant of $F(x, z)$, $G(y, z)$ as polynomials in z, a polynomial in x, y with coefficients in $\mathbf{C}$; this proves the theorem. //

In algebraic terms, these results mean that $\mathbf{E}(2\Omega) = \mathbf{C}(\wp(u|2\Omega), \wp'(u|2\Omega))$; that the transcendence of $\mathbf{E}(2\Omega)$ over $\mathbf{C}$ is unity; and that it is a quadratic extension of $\mathbf{C}(\wp(u|2\Omega))$, which itself is a transcendental extension of $\mathbf{C}$.

18. The Weierstrass function for a real lattice

For any lattice 2Ω we have, somewhat analogously to the homogeneity of $\wp u$, the relation $\wp(\bar{u}|2\bar{\Omega}) = \overline{\wp(u|2\Omega)}$; for simultaneously replacing u by $\bar{u}$ and each ω by $\bar{\omega}$ in the series defining $\wp u$, merely replaces every term in the series by its complex conjugate. In particular if the lattice 2Ω is real, $\wp(u|2\Omega)$ is a real function, i.e. one which assumes conjugate complex (possibly real and coincident) values for conjugate complex values of the variable, and, in particular, real values for real values of the variable. $\wp u$ is real, not only for real values of u, but for all values such that $u \pm \bar{u} \equiv 0 \pmod{2\Omega}$, since for all such values $\wp\bar{u} = \wp u$. This will be the case whenever either $\mathrm{Re}(u)$ or $i\mathrm{Im}(u)$ is in Ω. Thus in the u plane, the vertical lines cutting the real axis in all real periods and half periods, and the horizontal lines cutting the pure imaginary axis in all pure imaginary periods and half periods, are loci on which $\wp u$ has real values.

We have seen also that for real Ω, $S_n(2\Omega)$ is real for all n, so that in particular g_2, g_3 are real, and hence also all coefficients in the series (11.4) are real, being rational functions of g_2, g_3 with rational coefficients. In this series, for small values of u, the term $\frac{1}{u^2}$ dominates all the rest; thus when u is small and real, $\wp u$ is large, real, and positive; when u is small and pure imaginary, $\wp u$, is large, real, and negative. Put slightly differently, when u tends to zero through real or pure imaginary values, $\wp u$ tends through real values to $+\infty$ or $-\infty$ respectively.

19. Ω Rectangular

Take ω_1 real, ω_2 pure imaginary. $\wp u$ is real on all the sides of the rectangles forming the lattice Ω of half periods. As u describes continuously the perimeter of such a rectangle, say from $u = 0$ to $u = \omega_1$, then to $u = \omega_1 + \omega_2 = -\omega_3$, then to $u = \omega_2$, and finally back to $u = 0$, $\wp u$ varies continuously through exclusively real values, from $+\infty$ to $-\infty$; it must therefore assume every real value at least once in the course of the variation, and none more than once, since the path contains no two distinct values u, u' satisfying $u \pm u' \equiv 0 \pmod{2\Omega}$. The variation is thus a steady decrease of $\wp u$ through real values. As the contour round which u varies passes successively through the values $u = \frac{1}{2}\omega_1$, ω_1, $\omega_1 + \frac{1}{2}\omega_2$, $-\omega_3$, $\frac{1}{2}\omega_1 + \omega_2$, ω_2, $\frac{1}{2}\omega_2$, we have

$$e_1 + d_1 > e_1 > e_2 - d_2 > e_3 > e_1 - d_1 > e_2 > e_2 + d_2 ,$$

these values being all real. (As $e_1 > e_3 > e_2$, d_1^2, d_2^2 are both positive, the square roots d_1, d_2 as defined being positive and negative respectively; d_3^2 on the other hand is negative, so that $\wp(\frac{1}{2}\omega_1+\frac{1}{2}\omega_2) = e_3 + d_3$, $\wp(\frac{1}{2}\omega_1-\frac{1}{2}\omega_2) = e_3 - d_3$ are as we expect conjugate imaginary.) A further curious result is that if $Re(u)$ is half an odd multiple of ω_1, $|\wp u-e_1| = d_1$; for if $u - \frac{1}{2}k\omega_1$ is pure imaginary, where k is an odd integer, $u + u = k\omega_1$, i.e. $\bar{u} \equiv \omega_1 - u \pmod{2\Omega}$, and $|\wp u-e_1|^2 = (\wp u-e_1)(\wp\bar{u}-e_1) = d_1^2$. Similarly if $u-\frac{1}{2}k\omega_2$ is real, where again k is an odd integer, $|\wp u-e_2| = -d_2$, since $-\bar{u}\equiv\omega_2-u \pmod{2\Omega}$.

The loci of real values of $\wp u$ in the u plane are the horizontal and vertical lines through all the lattice points 2Ω, and also the horizontal and vertical lines half way between these. On the former horizontal lines the values of $\wp u$ oscillate between $+\infty$ and the minimum e_1, and on the latter horizontal lines they oscillate between the maximum e_3 and minimum e_2; on the former vertical lines the oscillation is between $-\infty$ and the maximum e_2, and on the latter between the maximum e_1 and the minimum e_3. It is to be noted that each stationary value is a maximum real value on either the horizontal or the vertical line through the stationary point, and a minimum on the other. Half way again between the loci of real values of $\wp u$ are the vertical lines on which $|\wp u-e_1|=d_1$, and the horizontal lines on which $|\wp u-e_2| = -d_2$. These two families of lines intersect in the four residue classes $\pm\frac{1}{2}\omega_1\pm\frac{1}{2}\omega_2+2\Omega$, in which $\wp u = e_3\pm d_3$; e_1, e_2, e_3 being real and d_3 pure imaginary,

$$|e_3 \pm d_3 - e_1| = d_1, \quad |e_3 \pm d_3 - e_2| = -d_2$$

reduce to the easily verified identity $d_i^2+d_j^2 = (e_i-e_j)^2$, where i, j are any two of 1, 2, 3.

20. Ω Rhombic

Take ω_1, ω_2 conjugate imaginary, ω_3 real and positive, and define as before $\omega_4 = \omega_1 - \omega_2$. $\wp u$ is real on the sides of the pattern of rectangles, with sides ω_3, ω_4, diagonals $2\omega_1$, $2\omega_2$, and vertices in the two residue classes 2Ω, $\omega_3 + 2\Omega$, whose mid-points are the residue classes $\omega_1 + 2\Omega$, $\omega_2 + 2\Omega$. The loci of real values are thus the horizontal and vertical lines through all the lattice points 2Ω, but not as in the rectangular case the lines half way between these. As u varies from $u = 0$ along the horizontal line to $u = \omega_3$, and then along the vertical to $u = -2\omega_1$ (downwards) or to $u = -2\omega_2$ (upwards), $\wp u$ passes through all real values, decreasing steadily from $+\infty$ to $-\infty$, through the value e_3 (which is accordingly real) at $u = \omega_3$; thus on the horizontal loci of real values, $\wp u$ oscillates between $+\infty$ and the minimum e_3, and on the vertical loci it oscillates between $-\infty$ and the maximum e_3. As e_3 is real and e_1, e_2 conjugate imaginary, d_3^2 is real and positive, and as $u = \frac{1}{2}\omega_3$ is in the horizontal part and either $u = \frac{1}{2}\omega_3 - \omega_1$ or $u = \frac{1}{2}\omega_3 - \omega_2$ in the vertical part of the path described, d_3 is the positive square root. If either $u - \frac{1}{2}k\omega_3$ is pure imaginary, or $u - \frac{1}{2}k\omega_4$ real, for any odd integer k, an exactly similar argument to that in the rectangular case shows that $|\wp u - e_3| = d_3$. The loci on which this is the case are, as before, the horizontal and vertical lines half way between those on which $\wp u$ is real; these intersect in the residue classes $\omega_1 + 2\Omega$, $\omega_2 + 2\Omega$, where $\wp u = e_1, e_2$; and in fact, as e_1, e_2 are conjugate imaginary and e_3 real, $|e_1 - e_3| = |e_2 - e_3| = d_3$.

It may be noted that whereas in the rectangular case we almost always take ω_1 real and ω_2 pure imaginary, $\mathrm{Re}(\omega_2/\omega_1) = 0$, in the rhombic case the choice of ω_1, ω_2 conjugate imaginary, ω_3 real, $|\omega_2/\omega_1| = 1$, is not the only one that is sometimes useful. For instance ω_1 real and ω_2, ω_3 conjugate imaginary makes $\mathrm{Re}(\omega_2/\omega_1) = -\frac{1}{2}$; or ω_1, $-\omega_2$ conjugate imaginary makes ω_3 pure imaginary, ω_4 real, and $|\omega_2/\omega_1| = 1$ as before.

21. The square and triangular lattices

We noted in Section 4 that if Ω is square $S_n(\Omega) = 0$ unless n is divisible by 4, and if Ω is triangular $S_n(\Omega) = 0$ unless n is divisible by 6. Thus for the square lattice $g_3 = 0$ and for the triangular lattice $g_2 = 0$. These conditions are mutually exclusive, as we have seen that g_2, g_3 cannot both vanish for any proper double lattice, and from the homogeneity of g_2, g_3 they are not only necessary but sufficient for the lattice to be square or triangular respectively.

For the square lattice e_1, e_2, e_3, being the roots of $4x^3-g_2x=0$, are one zero and the other two equal and opposite; in fact if $\omega_2=i\omega_1$, $\omega_3\equiv i\omega_3 \pmod{2\Omega}$, so that as $i\Omega=\Omega$, $\wp(i\omega_1)=-\wp\omega_1$, $\wp(i\omega_3)=-\wp\omega_3$, i. e. $e_2=-e_1$, $e_3=-e_3=0$. As ∞, $e_3=0$ separate e_1, $e_2=-e_1$ harmonically, the double values of any function of order 2 are a harmonic set; and for this reason the elliptic functions with a square lattice of periods are known as harmonic. In the rectangular position ω_1 is real and $\omega_2=i\omega_1$; in the rhombic position ω_3 is real and ω_1, $\omega_2=\frac{1}{2}(-1\pm i)\omega_3$. If Ω is the square lattice in either of these positions, $i^{\frac{1}{2}}\Omega$ is in the other position, and $\wp(i^{\frac{1}{2}}u\,|\,2i^{\frac{1}{2}}\Omega)=-i\wp(u\,|\,2\Omega)$. Thus as in the rectangular position $\wp u$ is real on the sides of the squares whose vertices are the lattice points Ω, in the rhombic position $\wp u$ is pure imaginary on these lines; similarly, as in the rhombic position $\wp u$ is real on one diagonal of each of these squares (that through the one vertex in the residue class 2Ω), in the rectangular position $\wp u$ is pure imaginary on these diagonals. Finally as e_1, e_2, the non-zero roots of $4x^3-g_2x=0$, are real for the rectangular and conjugate imaginary for the rhombic position, g_2 is positive for the former and negative for the latter, agreeing, incidentally, with the homogeneity relation $g_2(i^{\frac{1}{2}}\Omega)=-g_2(\Omega)$.

For the triangular lattice we take $\omega_1:\omega_2:\omega_3=1:\varepsilon:\varepsilon^2$. If two of these are conjugate imaginary and the third real, the lattice is in the vertical position; in the horizontal position, ω_1, $-\omega_2$ are conjugate imaginary, ω_3 pure imaginary, and ω_4 real. $e_1:e_2:e_3=1:\varepsilon:\varepsilon^2$, both as being the roots of $4x^3-g_3=0$, and also because $\wp(\varepsilon u)=\varepsilon\wp u$. As $(1-\varepsilon)(1-\varepsilon^2)=3$, $d_i^2=3e_i^2$ $(i=1, 2, 3)$. Since ∞, 1, ε, ε^2 are an equianharmonic tetrad, so are the four double values of any function of order 2; and the elliptic functions with triangular lattice of periods are known as equianharmonic. The rhombus consists of two equilateral triangles with a common side, the shorter diagonal ω_3 of the rhombus being equal in length to its sides ω_1, ω_2. The centres of these triangles are in the residue classes $\pm\frac{2}{3}\omega_4+2\Omega$; and we note that as $(\omega_2-\omega_3)-(\omega_1-\omega_2)=3\omega_2$, $\omega_4\equiv\varepsilon\omega_4\equiv\varepsilon^2\omega_4 \pmod{3\Omega}$, i. e. $\frac{2}{3}\omega_4\equiv\frac{2}{3}\varepsilon\omega_4\equiv\varepsilon^2\omega_4 \pmod{2\Omega}$, and $\wp(\frac{2}{3}\omega_4)=\varepsilon\wp(\frac{2}{3}\omega_4)=0$, i. e. the zeros of $\wp u$ are the centres of the two sets of triangles. These are also stationary points of $\wp'u$, at which $\wp''u = \wp'''u = 0$, since they are zeros of $P_n(u)$ except when n is divisible by 3, by the same argument as for $\wp u$. In fact the three residue classes in which $\wp'u$ assumes any given value are of the form $v + 2\Omega$, $\varepsilon v + 2\Omega$, $\varepsilon^2 v + 2\Omega$; if two of these coincide, all three do, and this occurs in the three residue classes $v = 0, \pm\frac{2}{3}\omega_4$.

22. Invariants of the real lattice

We have seen that g_2, g_3 are both real for a real lattice (more generally $g_i(\bar{\Omega}) = \overline{g_i(\Omega)}$, $i = 2, 3$). Conversely, if g_2, g_3 are both real, so is the lattice, since (11.4) defines a real function, and if ω is a period of this, so is $\bar{\omega}$. The discriminant of the equation $4x^3 - g_2x - g_3 = 0$ is $\Delta = g_2^3 - 27g_3^2$. This means in the first place that two roots (at least) of the equation coincide if and only if $g_2^3 = 27g_3^2$; but also, for real g_2, g_3, that the roots are all real or one real and two conjugate imaginary, according as $g_2^3 >$ or $< 27g_3^2$. Thus for real g_2, g_3, $g_2^3 > 27g_3^2$ is the necessary and sufficient condition for a rectangular, and $g_2^3 < 27g_3^2$ that for a rhombic lattice; and $g_2^3 \neq 27g_3^2$ for any proper double lattice, since we have seen that in all cases e_1, e_2, e_3 are all distinct.

It is convenient to classify rhombic lattice shapes (other than the square and the triangular) into extreme and medium, according as the shorter diagonal of the rhombus is shorter or longer than its sides, i. e. according as the ratio of the lengths of the diagonals is greater than $\sqrt{3}$, or between $\sqrt{3}$ and 1; the limiting values $\sqrt{3}$, 1 of this ratio corresponding evidently to the triangular and square shapes.

Theorem 2. 9. For the real lattice, whether rectangular or rhombic, g_3 is positive or negative according as the lattice is vertical or horizontal. g_2 is positive for all real rectangular lattices, and for the real rhombic lattice g_2 is positive or negative according as the lattice is extreme or medium.

Proof. Consider a variable lattice Ω, generated by ω_1 (constant and real) and variable $\omega_2 = \tau\omega_1$. We obtain one sample of every rectangular lattice shape, in the vertical position, by keeping $\mathrm{Re}(\tau) = 0$

constant, and letting $\text{Im}(\tau)$ increase steadily from 1 (square lattice) to $+\infty$ (the degenerate limiting shape). In the same way we obtain one sample of every rhombic lattice shape, in the vertical position, by keeping $\text{Re}(\tau) = -\frac{1}{2}$ constant, and letting $\text{Im}(\tau)$ increase steadily from $\frac{1}{2}$ (square lattice), through $\frac{1}{2}\sqrt{3}$ (triangular lattice), to $+\infty$ (again the degenerate limiting shape), so that $\frac{1}{2} < \text{Im}(\tau) < \frac{1}{2}\sqrt{3}$ gives the medium and $\text{Im}(\tau) > \frac{1}{2}\sqrt{3}$ the extreme rhombic shapes. By Theorem 1.2, as $\text{Im}(\tau)$ tends to infinity, g_2, g_3 (for the period lattice 2Ω) tend, independently of $\text{Re}(\tau)$, to the limits $g_2^* = \frac{15}{2} s_4 \omega_1^{-4}$, $g_3^* = \frac{35}{8} s_6 \omega_1^{-6}$, both obviously positive. Thus in both the prescribed variation processes, with $\text{Re}(\tau) = 0$ and with $\text{Re}(\tau) = -\frac{1}{2}$, g_3 starts from the value 0 for the square lattice, and varies continuously remaining real and not passing again through the value 0, and tending ultimately to the positive limit g_3^*; it thus remains positive throughout the process; i.e. g_3 is positive for all vertical real lattices, whether rectangular or rhombic; and since if Ω is vertical $i\Omega$ is horizontal, and $g_3(i\Omega) = -g_3(\Omega)$, g_3 is negative for all horizontal lattices, rectangular or rhombic. As for g_2, it is the same for the vertical and horizontal lattices, as $g_2(i\Omega) = g_2(\Omega)$; it is positive for all rectangular lattices, since $g_2^3 > 27g_3^2 > 0$; and for the rhombic lattices, as $\text{Im}(\tau)$ varies from $\frac{1}{2}$ through $\frac{1}{2}\sqrt{3}$ to $+\infty$, with $\text{Re}(\tau) = -\frac{1}{2}$, g_2 starts from a negative value (for the square lattice in the rhombic position), passes through the value 0 only at $\text{Im}(\tau) = \frac{1}{2}\sqrt{3}$ (for the triangular lattice), and tends ultimately to the positive limit g_2^*; it is thus negative for $\frac{1}{2} < \text{Im}(\tau) < \frac{1}{2}\sqrt{3}$, the medium rhombic shapes, and positive for $\text{Im}(\tau) > \frac{1}{2}\sqrt{3}$, the extreme rhombic shapes; and the theorem is proved. //

A simple corollary from this is that the zeros of $\wp u$ are in the longer symmetry axis of the fundamental unit cell, whether this is rectangular or rhombic. For the real rectangular lattice Ω, as $e_1 > e_3 > e_2$ and $e_1 + e_2 + e_3 = 0$, $e_1 > 0 > e_2$, and the sign of e_3 is opposite to that of $g_3 = 4e_1e_2e_3$; thus according as Ω is vertical or horizontal, 0 is between e_1, e_3 or between e_2, e_3. Similarly, for the real rhombic lattice Ω, as e_1, e_2 are conjugate imaginary, e_3 has the same sign as g_3, i.e. 0 is $< e_3$ or $> e_3$ according as Ω is vertical or horizontal.

Theorem 2.10. The limiting values g_2^*, g_3^* of the invariants, corresponding to the simple lattice with real generator ω_1 as degenerate limit of a variable double lattice, satisfy $g_2^{*3} = 27g_3^{*2}$.

Proof. In the variation process with $\mathrm{Re}(\tau) = 0$ described in the proof of the last theorem, $g_2^3 > 27g_3^2$ at all stages; hence in the limit $g_2^{*3} \geq 27g_3^{*2}$. Similarly in the variation with $\mathrm{Re}(\tau) = -\frac{1}{2}$, $g_2^3 < 27g_3^2$ at all stages, so that $g_2^{*3} \leq 27g_3^{*2}$. //

This means that there is a positive real number e^* such that $g_2^* = 3e^{*2}$, $g_3^* = e^{*3}$, and $4x^3 - g_2^*x - g_3^* = (2x+e^*)^2(x-e^*)$; thus the limits of e_1, e_2, e_3 are e^*, $-\frac{1}{2}e^*$, $-\frac{1}{2}e^*$. Similarly for the limiting form of the horizontal lattice (rectangular or rhombic), the simple lattice generated by $i\omega_1$, they are $\frac{1}{2}e^*$, $\frac{1}{2}e^*$, $-e^*$. We have also $s_4 = \frac{2}{5}e^{*2}\omega_1^4$, $s_6 = \frac{8}{35}e^{*3}\omega_1^6$; and we may round off this result by anticipating what will be proved when we come to study the Jacobi functions, that for $\omega_1 = \frac{1}{2}\pi$, $e^* = \frac{2}{3}$; thus the actual values of s_4, s_6 are $s_4 = \frac{\pi^4}{90}$, $s_6 = \frac{\pi^6}{945}$. These values are also known of course from the theory of the Riemann zeta function.

23. Properties of $\wp'u$ for the real lattice

Evidently $\wp'u$ is real on all horizontal lines in the u plane on which $\wp u$ is real, and pure imaginary on all vertical lines on which $\wp u$ is real; put crudely, $d\wp u$ is real, and du real and pure imaginary respectively. But there are other loci of real values of $\wp'u$ in the u plane, which meet the horizontal lines at angles of $\pm\frac{\pi}{3}$ in the triple poles (the lattice points 2Ω), and cross them orthogonally in any real stationary points of $\wp'u$. As $\wp''u = 6\wp^2u - \frac{1}{2}g_2$, $\wp'u$ is stationary in the four residue classes in which $\wp^2u = \frac{g_2}{12}$ (only when Ω is triangular, these coincide by pairs in the zeros of $\wp u$, the residue classes $\pm\frac{2}{3}\omega_4$). The corresponding stationary values are obtained by eliminating $x = \wp u$ between $4x^3 - g_2x - (\wp'^2u+g_3) = 0$ and $12x^2 - g_2 = 0$; but this simply gives the discriminant of $4x^3 - g_2x - (\wp'^2u+g_3) = 0$, namely $g_2^3 - 27(\wp'^2u+g_3)^2 = 0$, so that the stationary values of $\wp'u$ are

$$\pm\{-g_3 \pm (\tfrac{g_2}{3})^{3/2}\}^{\frac{1}{2}}.$$

For the rectangular lattice, the roots of $4x^3 - g_2x - g_3 = 0$ being all real, those of its derivative $12x^2 - g_2 = 0$ are real, one between e_1, e_3 and one between e_2, e_3. Thus the stationary points of $\wp'u$ are two in each symmetry axis of the rectangular unit cell, those in the horizontal axis giving real and those in the other pure imaginary stationary values. As the values of $\wp u$ on this horizontal line oscillate between the maximum e_3 and minimum e_2, those of $\wp'u$ likewise oscillate between a real maximum and minimum, which occur alternately between consecutive zeros of $\wp'u$, the maxima and minima of $\wp u$.

For the extreme rhombic lattice $g_2 > 0$, so that the stationary points of $\wp'u$ all occur at points where $\wp u$ is real, i.e. in the diagonals of the rhombic unit cell. As $|g_3| > |\frac{g_2}{3}|^{3/2}$, the stationary values are all real or all pure imaginary, according as $g_3 <$ or > 0, i.e. the stationary points are in the longer diagonal of the rhombus, and are real if the lattice is horizontal. Thus for the horizontal extreme rhombic lattice (and no other) $\wp'u$, instead of increasing steadily along the real axis from $-\infty$ at $u = 0$ to $+\infty$ at $u = 2\omega_1$, reaches a negative maximum, decreases to a negative minimum, increases again through 0 at $u = \omega_1$ to a positive maximum, decreases to a positive minimum, and then increases again. The graph of $\wp u$ over the same range, instead of being (as in other cases) everywhere concave upwards, has on each side of its vertical symmetry axis at $u = \omega_1$ a pair of points of inflexion, the arc between which is concave downwards; in the triangular limiting case each of these pairs coincides, at $u = \frac{2}{3}\omega_3, \frac{4}{3}\omega_3$, $\wp u = 0$, in a point of undulation of the graph, i.e. one where the tangent meets it in four consecutive points.

For the medium rhombic lattice the roots of $12x^2 - g_2 = 0$ are imaginary, so that the stationary points of $\wp'u$ are two equal and opposite pairs of conjugate imaginary residue classes, i.e. of the form $\pm a \pm ib + 2\Omega$; and the stationary values are likewise two equal and opposite pairs of conjugate imaginaries.

24.

Figures 1, ..., 6 show a unit cell, with one vertex at the origin, for a typical rectangular lattice, the square lattice in the rectangular and rhombic positions, a medium rhombic, the triangular, and an extreme rhombic lattice. In the left hand half of each, the loci of real values of $\wp u$ are shown as continuous and those of its pure imaginary values as broken lines, and in the right hand half the same loci for $\wp' u$. Note that at the double poles of $\wp u$ (and at its double zeros in the square case) two loci of real and two of pure imaginary values intersect symmetrically, and at the triple poles of $\wp' u$ three of each. At a stationary point where the value is real or pure imaginary, two loci of real or pure imaginary values intersect orthogonally, and at the real triple values of $\wp' u$ in the triangular case three loci of real values intersect at equal angles. Other stationary points, where the value is neither real nor pure imaginary, are shown as ringed dots.

These loci divide the plane into regions, each of which is a map of one quadrant of the x plane, where $x = \wp u$ or $\wp' u$. These are numbered 1, 2, 3, 4, with the usual convention for the quadrants: 1, $0 < \theta < \frac{\pi}{2}$; 2, $\frac{\pi}{2} < \theta < \pi$; 3, $\pi < \theta < \frac{3\pi}{2}$; 4, $\frac{3\pi}{2} < \theta < 2\pi$; where $\theta = \mathrm{Im}(\log x)$. Every value of x in the quadrant is assumed once in the region, except that where a stationary point is interior to the region, every value in the quadrant is assumed at two points in the region, which are (precisely in the case of $\wp u$, roughly in that of $\wp' u$) symmetrically placed on either side of the stationary point.

The rectangular lattice in Figure 1 is that for which the zeros of $\wp u$ are the residue classes $\pm\frac{1}{2}\omega_1 + \omega_2 + 2\Omega$, so that $d_1 = e_1$. From $e_1^2 = (e_1 - e_2)(e_1 - e_3)$, $e_1 + e_2 + e_3 = 0$, we trivially obtain $e_1 : e_2 : e_3 = 1 : \frac{1}{2}(-1-\sqrt{5}) : \frac{1}{2}(-1+\sqrt{5})$, $k' = \frac{1}{2}(3-\sqrt{5})$, $g_2^3 = 32 g_3^2$.

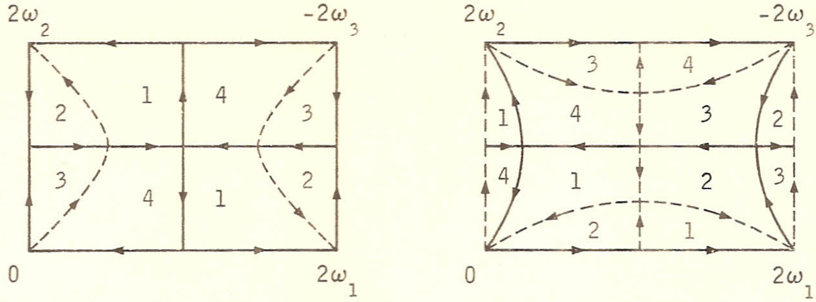

Figure 1

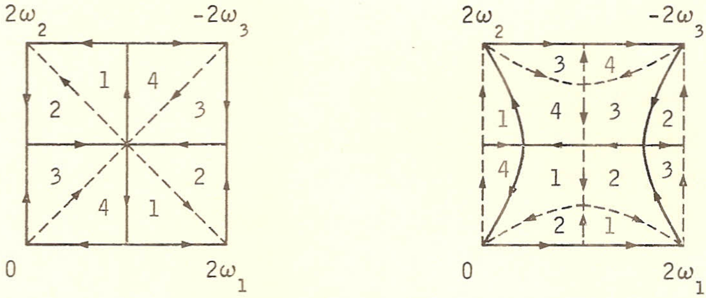

Figure 2

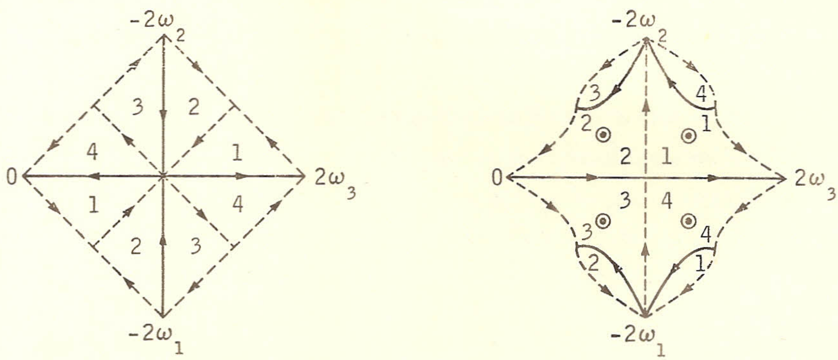

Figure 3

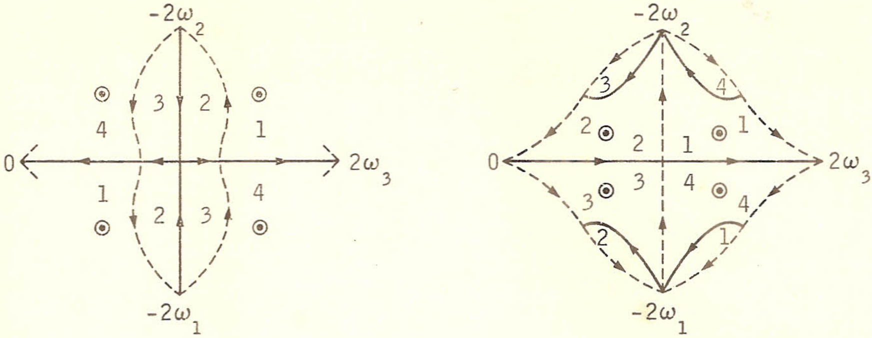

Figure 4

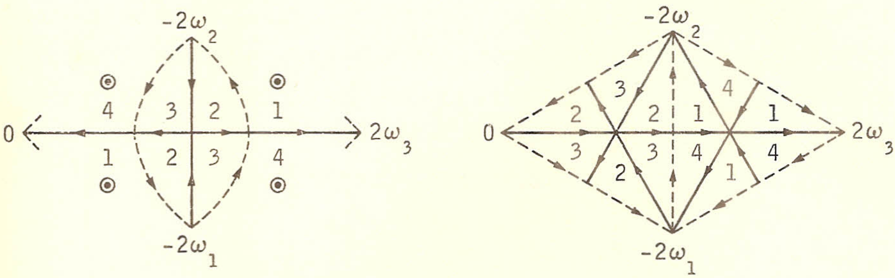

Figure 5

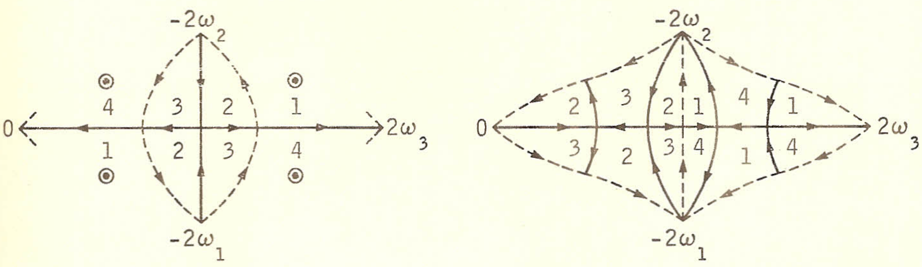

Figure 6

Historical Note

What are now called elliptic integrals began to be studied in the mid seventeenth century, in connexion with the rectification of the ellipse and other curves; but what was described by Jacobi as the birthday of the subject was Euler's discovery in 1751 of Fagnano's work [13], which stimulated him to a sequence of papers that fill two volumes of his Omnia Opera [12]. This study of the integrals culminated in Legendre's two great works [26, 27]. Gauss, as early as 1797, had inverted an integral, i. e. defined the upper limit as a function of the value of the integral instead of the other way about, and so produced the first true elliptic functions, though only for the square lattice; but his work [15] remained unpublished and unknown till long afterwards.

The inversion of the integral as a general process, and the definition of elliptic functions for any modulus, is due to Abel [1] and Jacobi [22] in 1827, independently but soon in cooperation, just in time to be referred to enthusiastically by Legendre in the final volume of [27]; this culminated in Jacobi's Fundamenta Nova [23] in 1829, in which he also introduced the first theta functions Θu, Hu. His later notation for the theta functions, together with the use of these to define the elliptic functions, was introduced in a lecture [24] in 1838.

In the next half century countless papers and some substantial books were published, elaborating and multiplying the identities and other formulae, but adding comparatively little to the basic theory; though Liouville's lectures [28] in 1847 added the powerful tool of his theorem; Riemann's introduction about 1855 [29, 31] of the multiple surfaces named after him greatly clarified the process of integration; and significant improvements in notation (as well as much else) were introduced by Hermite [20] (the classification of theta functions by their characteristic) in 1858, and by Gudermann [18] in 1838 and Glaisher [16] in 1881 (cf. Section 31). The text books of this period, such as Verhulst [39], Briot and Bouquet [3], and Enneper [11], mostly used the theta functions to define the elliptic functions; but Cayley [4] in 1876 went back to the Fundamenta Nova, defining the functions by inverting the integrals, and using no theta functions except Θu, Hu. (This is far the most valuable account of the classical theory for English readers; the chapters on the addition theorem, and on transformations, are well worth some study, if only to make us realise the vast difference between the classical and modern approaches to the whole subject.)

Weierstrass in lectures from 1862 onwards [42, 34] defined the functions $\wp u$, ζu, σu, $\sigma_j u$ $(j = 1, 2, 3)$; but the use of the theta functions as the logical starting point still seems to have had such a hold on everyone's mind, that both he and the major text book writers who followed him, such as Weber [41] in 1891, Tannery and Molk [37] in 1893, and Appel and Lacour [2] in 1896, defined σu directly, and obtained ζu, $\wp u$ by differentiating $\log \sigma u$, and the Jacobi functions as quotients of sigma functions; and Copson [5] as late as 1935 uses substantially the same approach. Greenhill [17] on the other hand, in 1893, followed Cayley in general treatment, but brought in the Weierstrass functions rather as a sideline, defining $\wp u$, like the other functions, by inverting the appropriate integral. (This book is rich in formulae of all kinds, and also in geometrical and mechanical applications of the theory.)

A fuller account of the history of the subject, down to about this point, together with a much fuller bibliography, is to be found in Fricke's article [14] in the Encyklopädie der Mathematischen Wissenschaften.

Expressions for the invariants, such as k, K, and later J, in terms of q or τ, and some study of the effect on these of the transformations of the modular group, are of course to be found in many of the above authors; but the whole theory of the modular pattern, the modular group and its subgroups of finite index, and the application of these to the modular relations for transformations of various orders, at any rate in anything like its modern form, is essentially the creation of Klein, whose monumental work [25] appeared in 1890. (At an earlier date the term modular function seems to have been applied to the elliptic functions themselves, to judge from the title of Gudermann's paper [18].) I may add that I have myself devoted some study to the singularities that occur in the modular relations of low orders [7, 8].

The direct definition of $\wp u$ as a double series, as the starting point of the whole subject, is found for the first time, as far as I can discover, in Whittaker and Watson [43] in 1902; their treatment of the Weierstrass functions has been followed in essence in these notes, though with many shifts in emphasis. They still, however, thought it necessary to deal with the theta functions before the Jacobi functions, and to define the latter in terms of the former; with the result that the Weierstrass and Jacobi functions appear as two almost unrelated topics. It is to be added that their work is illustrated by a great wealth of examples and exercises, of every degree of difficulty.

The definition of the Jacobi functions as root functions, from the Weierstrass function, which is the basis of our treatment of them, and of our whole approach to the classical theory, is due to Neville [30] in 1944, though of course the expressions for the squares of the Jacobi functions in terms of the Weierstrass function were familiar. His book is imaginative, and extremely valuable, though rather diffuse, and not altogether easy to read; and he seems to have missed the easy derivation of the addition theorem for the Jacobi functions, by elementary algebra, from that for the Weierstrass function.

The only other substantial work on elliptic functions that has appeared, so far as I know, since early this century, is that of Eagle [10] in 1958. He defines all the functions he uses by trigonometric series, with the result that his fundamental lattice is Ω_π throughout; and as he gives new names and notation to everything, it is not always easy to compare his work with that of other writers. In spite of this, and of some elementary blunders, his book is not without interest.

The ternary functions are, so far as I know, my own invention, and first saw print, very much as they appear here, in 1964 [6].

The only useful tables of elliptic functions known to me are the Smithsonian Tables [36], which give to 12 places of decimals the values of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, $E(u)$, for real values of u at intervals of $\frac{K}{90}$, for all moduli $k = \sin\theta$, where θ is a whole number of degrees.

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Definitions and notation introduced ad hoc for a particular context only are not indexed, but it is hoped that all those which, once introduced, are reused later, have been included. Greek letters are listed separately, after the Latin-alphabet index.

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