Problem Set 6 Solutions

March 31, 2005

3.2.1

(a) First consider the problem

$$\min x_1 + x_2$$

$$\text{subject to } x_1^2 + x_2^2 = 2.$$

Note that $\nabla h(x) = 2x \neq 0$ for all feasible x. Thus any feasible x is regular, and we can apply the Lagrange Multiplier Theorem. We have

$$L(x^*, \lambda^*) = x_1^* + x_2^* + \lambda^*((x_1^*)^2 + (x_2^*)^2 - 2).$$

If x^* is a local minimum,

$$\nabla_x L(x^*, \lambda^*) = 0 \quad \text{or} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda^* \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 0,$$

and

$$\nabla_\lambda L(x^*, \lambda^*) = 0 \quad \text{or} \quad (x_1^*)^2 + (x_2^*)^2 = 2.$$

Combining these two equations, we see that the only possible candidates for being a local minimum are $(1, 1)$ with $\lambda^* = -1/2$ and $(-1, -1)$ with $\lambda^* = 1/2$.

We also have

$$\nabla^2_{xx} L(x^*, \lambda^*) = 2\lambda^* I.$$

From the second order necessary conditions, it follows that $(1, 1)$ cannot be a local minimum, while $(-1, -1)$ is a strict local minimum. Since $f(x)$ is continuous over the constraint set, which is compact, a global minimum exists. Thus $(-1, -1)$ is the unique global minimum. Now consider the problem

$$\min -(x_1 + x_2)$$

$$\text{subject to } x_1^2 + x_2^2 = 2.$$

Using an argument analogous to that above, we obtain that $(1, 1)$ is the unique global minimum for this new problem or, equivalently, that $(1, 1)$ is the unique global maximum of the original problem.

(b) Consider the problem

$$\min x_1 + x_2$$

$$\text{subject to } x_1^2 + x_2^2 - 2 = u.$$

The only change from part (a) is

$$\nabla_\lambda L(x^*, \lambda^*) = 0 \quad \text{or} \quad (x_1^*)^2 + (x_2^*)^2 - 2 = u.$$

Combining the preceding relation with $\nabla_x L(x^*, \lambda^*) = 0$, we have for $u > -2$

$$x(u) = (\sqrt{(2+u)/2}, \sqrt{(2+u)/2}) \text{ with } \lambda(u) = -1/(\sqrt{2(2+u)})$$

or

$$x(u) = (-\sqrt{(2+u)/2}, -\sqrt{(2+u)/2}) \text{ with } \lambda(u) = 1/(\sqrt{2(2+u)}).$$

Using the second order necessary conditions, we see that the first point is not a local minimum, while the second point is a strict local minimum. Substituting the second solution $x(u)$ into the cost function, we obtain

$$p(u) = f(x(u)) = -2\sqrt{(2+u)/2} = -\sqrt{2(2+u)},$$

and

$$\nabla p(u) = -\frac{1}{\sqrt{2(2+u)}} = -\lambda(u), \quad u > -2.$$

If we let S be the set $\{u \in | \, |u| < 2\}$ (an open sphere centered at $u = 0$), then $\nabla p(u) = -l(u)$ for all $u \in S$. Hence the gradient of the primal function is related to the Lagrange multiplier as specified by the sensitivity theorem.

3.2.3

a) The problem is to maximize xyz subject to $x + y + z = a$, where a is a given positive number. If (x^*, y^*, z^*) is an optimal solution, we clearly have $0 < x^*$, $0 < y^*$, $0 < z^*$, and that (since all feasible points are regular) there exists λ^* such that

$$x^* y^* = z^* y^* = x^* z^* = -\lambda^*.$$

Thus $x^* = y^* = z^* = a/3$ is the only solution to the 1st order necessary conditions. Since the problem is equivalent to maximizing xyz subject to the compact set constraints $0 \leq x \leq a$, $0 \leq y \leq a$, $0 \leq z \leq a$, and $x + y + z = a$, there exists a global maximum, which must be the only positive solution $(a/3, a/3, a/3)$ of the 1st order optimality conditions.

We may also check that the 2nd order sufficiency conditions for a local maximum are satisfied at $x^* = y^* = z^* = a/3$. The Hessian of the Lagrangian at that point is

$$\begin{pmatrix} 0 & -a/3 & -a/3 \\ -a/3 & 0 & -a/3 \\ -a/3 & -a/3 & 0 \end{pmatrix},$$

which is seen to satisfy the 2nd order sufficiency condition by using an argument that is identical to the corresponding argument of Example 3.2.1 in Section 3.2.

(b) Similar to part (a), the optimal sides are $x^* = y^* = z^* = V^{1/3}$ where V is the given volume.

3.2.4

(a) Without loss of generality, assume that $j = 1$. We have

$$\nabla f(x^*) + \lambda_1^* \nabla h_1(x^*) + \sum_{i=2}^{m} \lambda_i^* \nabla h_i(x^*) = 0, \tag{1}$$

$$y' \left(\nabla^2 f(x^*) + \lambda_1^* \nabla^2 h_1(x^*) + \sum_{i=2}^{m} \lambda_i^* \nabla^2 h_i(x^*) \right) y > 0, \tag{2}$$

for all $y \neq 0$ with $\nabla h_i(x^*)' y = 0$ for $i = 1, \ldots, m$. Divide Eq. (1) by λ_1^*. Then x^* is feasible for the new optimization problem

$$\min h_1(x)$$

$$\text{subject to } \ f(x) = f(x^*), \ h_i(x) = 0, \ i = 2, \ldots, m$$

and $1/\lambda_1^*, \ \lambda_i^*/\lambda_1^*, \ i = 2, \ldots, m$ are Lagrange multipliers corresponding, respectively, to the equality constraints $f(x) = f(x^*), \ h_i(x) = 0, \ i = 2, \ldots, m$. Using Eq. (1) and the fact that $\lambda_1^* \neq 0$, it is easy to see that

$$\{y \mid \nabla h_i(x^*)' y = 0, \ \ \forall\, i\} = \{y \mid \nabla f(x^*)' y = 0, \ \nabla h_i(x^*)' y = 0, \ i = 2, \ldots, m\}. \tag{3}$$

Dividing Eq. (2) by λ_1^* and using Eq. (3), we have

$$y' \left(\nabla^2 h_1(x^*) + \frac{1}{\lambda_1^*} \nabla^2 f(x^*) + \sum_{i=2}^{m} \frac{\lambda_i^*}{\lambda_1^*} \nabla^2 h_i(x^*) \right) y > 0,$$

for all $y \neq 0$ with $\nabla f(x^*)' y = 0$, $\nabla h_i(x^*)' y = 0$ for $i = 2, \ldots, m$. Since x^* and $1/\lambda_1^*, \lambda_i^*/\lambda_1^*, \ i = 2, \ldots, m$ satisfy the sufficiency conditions for the new problem, we have that x^* is a local minimum of $h_1(x)$ subject to the given constraints.

(b) Here the only difference is that we divide Eq. (1) and Eq. (2) by $-1/\lambda_1^*$. Therefore x^* and $\nu_1^* = -1/\lambda_1^*, \ \nu_i^* = -\lambda_i^*/\lambda_1^*, \ i = 2, \ldots, m$, satisfy

$$-\nabla h_1(x^*) + \nu_1^* \nabla f(x^*) + \sum_{i=2}^{m} \nu_i^* \nabla h_i(x^*) = 0,$$

$$y' \left(-\nabla^2 h_1(x^*) + \nu_1^* \nabla^2 f(x^*) + \sum_{i=2}^{m} \nu_i^* \nabla^2 h_i(x^*) \right) y > 0,$$

for all $y \neq 0$ with $\nabla f(x^*)' y = 0$, $h_i(x^*)' y = 0$ for $i = 2, \ldots, m$. Therefore x^* is a local minimum of $-h_1(x)$ [i.e. a local maximum for $h_1(x)$] subject to the given constraints.

3.3.1

Since $\nabla^2 f(x,y)$ is positive definite and the constraints are linear, the necessary conditions for optimality are also sufficient. We have

$$\nabla f(x,y) = \begin{pmatrix} 2(x-a)+y \\ 2(y-b)+x \end{pmatrix}.$$

Let $g_1(x,y) = -x$, $g_2(x,y) = x-1$, $g_3(x,y) = -y$, $g_4(x,y) = y-1$ and ${}^*_j$ for $j = 1,2,3,4$ be corresponding Lagrangian multipliers. Then the necessary and sufficient condition for (x^*, y^*) to be optimal is

$$\begin{pmatrix} 2(x^*-a)+y^* \\ 2(y^*-b)+x^* \end{pmatrix} + \begin{pmatrix} \mu_2^* - \mu_1^* \\ \mu_4^* - \mu_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1}$$

for some Lagrange multipliers $\mu_i^* \geq 0$.

Case (i): Let $\nabla f(x^*, y^*) = 0$ with $0 < x^* < 1$ and $0 < y^* < 1$. Then the optimal solution is $x^* = \frac{2}{3}(2a-b)$, $y^* = \frac{2}{3}(2b-a)$ for $0 < 2a-b < \frac{3}{2}$, $0 < 2b-a < \frac{3}{2}$. In this case the optimality conditions are satisfied with all $\mu_j^* = 0$.

Case (ii): For $0 < b < 1$, $b - 2a > 0$, the point $(x^*, y^*) = (0, b)$ together with $\mu_1^* = b - 2a$, $\mu_2^* = \mu_3^* = \mu_4^* = 0$ satisfies the optimality conditions. For $a < 0$, $b < 0$ the optimal point is (0,0) and a possible choice for Lagrange multipliers can be $\mu_1^* = -2a$, $\mu_3^* = -2b$, $\mu_2^* = \mu_4^* = 0$. For $b > 1$ and $1 - 2a > 0$ the point $(0,1)$ is optimal. In this case, we can take Lagrange multipliers in (1) to be $\mu_4^* = 2(b-1)$, $\mu_1^* = 1 - 2a$, $\mu_2^* = \mu_3^* = 0$.

Case (iii): For $0 < b - \frac{1}{2} < 1$, $2a - b - \frac{3}{2} > 0$ the optimal point is $(1, b - \frac{1}{2})$, which satisfies the optimality conditions with ${}^*_2 = 2a - b - \frac{3}{2}$, $\mu_1^* = \mu_3^* = \mu_4^* = 0$. When $2b - 1 < 0$ and $a > 1$, the point $(1,0)$ together with $\mu_2^* = 2(a-1)$, $\mu_3^* = 1 - 2b$, $\mu_1^* = \mu_4^* = 0$ satisfies condition (1). Therefore, it is optimal. If $2b - 3 > 0$, $2a - 3 > 0$, then $(x^*, y^*) = (1,1)$ and $\mu_2^* = 2a-3$, $\mu_4^* = 2b-3$, $\mu_1^* = \mu_3^* = 0$ satisfy the condition (1), i.e., the point $(1,1)$ is a solution.

Case (iv): When $0 < a < 1$ and $b < 0$, the point $(a, 0)$ with $\mu_1^* = \mu_2^* = \mu_4^* = 0$, $\mu_3 = -2b$ satisfies the condition (1). Hence, it is an optimal point.

Case (v): For $0 < a - \frac{1}{2} < 1$ and $2b - a - \frac{3}{2} > 0$, the point $(a - \frac{1}{2}, 1)$ is optimal, because it satisfies the condition (1) with $\mu_1^* = \mu_2^* = \mu_3^* = 0$ and $\mu_4^* = 2b - a - \frac{3}{2}$.

3.3.2

Let us convert the given problem to the minimization problem

$$\text{minimize} \quad f(x) = -y'x$$

$$\text{subject to} \quad g(x) = x'Qx - 1 \leq 0.$$

The constraint set is compact and the cost function is continuous. Thus a global minimum exists.

First consider the case where $y = 0$. Then $f(x) = 0$ for any x satisfying $g(x) \leq 0$, and the desired result holds.

Now consider the case where $y \neq 0$. We have

$$\nabla f(x) = -y, \qquad \nabla g(x) = 2Qx.$$

Since Q is positive definite, any feasible point is regular. Assume that x^* is a local minimum. Then from the Kuhn-Tucker necessary conditions, there exists $\mu^* \geq 0$ such that

$$-y + \mu^* 2Qx^* = 0, \quad \text{and} \quad \mu^* = 0 \text{ if } x^{*\prime}Qx^* < 1.$$

Since $y \neq 0$, we must have $\mu^* > 0$ and thus $x^{*\prime}Qx^* = 1$. Pre-multiplying $y = \mu^* 2Qx^*$ by $x^{*\prime}$ yields

$$x^{*\prime}y = 2\mu^* x^{*\prime}Qx^* = 2\mu^*.$$

We thus have

$$x^* = \frac{Q^{-1}y}{2\mu^*} = \frac{Q^{-1}y}{x^{*\prime}y}$$

or

$$y'x^* = \frac{y'Q^{-1}y}{x^{*\prime}y}$$

or

$$x^{*\prime}y = \pm\sqrt{y'Q^{-1}y}.$$

The optimal value of the minimization problem is $-\sqrt{y'Q^{-1}y}$, and so the optimal value of the original problem is $\sqrt{y'Q^{-1}y}$.

Now, for any $x \neq 0$, let $\bar{x} = x/\sqrt{x'Qx}$. We have $\bar{x}'Q\bar{x} = 1$, and so from above

$$(\bar{x}'y)^2 \leq y'Q^{-1}y,$$

or equivalently

$$(x'y)^2 \leq (x'Qx)(y'Q^{-1}y).$$