

London Mathematical Society  
Lecture Note Series 110

---

An Introduction to  
the Theory of  
**Surreal Numbers**

HARRY GONSHOR

Published by the Press Syndicate of the University of Cambridge  
The Pitt Building, Trumpington Street, Cambridge CB2 1RP  
32 East 57th Street, New York, NY 10022, USA  
10, Stamford Road, Oakleigh, Melbourne 3166, Australia

© Cambridge University Press 1986

First published 1986

Printed in Great Britain at the University Press, Cambridge

Library of Congress cataloging in publication data

Gonshor, H.  
An introduction to the theory of surreal numbers (London  
Mathematical Society lecture note series; 110)  
Bibliography: p.  
1. Numbers, Theory of. I. Title. II. Series  
QA241.G63 1986 512'.7 86-9668

British Library cataloguing in publication data

Gonshor, H.  
An introduction to the theory of surreal numbers - (London  
Mathematical Society lecture note series, ISSN 0076-0052; 110)  
1. Numbers, Theory of, 2. Logic  
I. Title II. Series  
512'.7 QA241

ISBN 0 521 31205 1

CONTENTS

	<u>page</u>
<a href="#">Preface</a>	
<a href="#">Acknowledgements</a>	
<a href="#">Chapter 1 Introduction</a>	<a href="#">1</a>
<a href="#">Chapter 2 Definition and Fundamental Existence Theorem</a>	<a href="#">3</a>
A. <a href="#">Definition</a>	<a href="#">3</a>
B. <a href="#">Fundamental Existence Theorem</a>	<a href="#">4</a>
C. <a href="#">Order Properties</a>	<a href="#">8</a>
<a href="#">Chapter 3 The Basic Operations</a>	<a href="#">13</a>
A. <a href="#">Addition</a>	<a href="#">13</a>
B. <a href="#">Multiplication</a>	<a href="#">17</a>
C. <a href="#">Division</a>	<a href="#">21</a>
D. <a href="#">Square Root</a>	<a href="#">24</a>
<a href="#">Chapter 4 Real Numbers and Ordinals</a>	<a href="#">27</a>
A. <a href="#">Integers</a>	<a href="#">27</a>
B. <a href="#">Dyadic Fractions</a>	<a href="#">28</a>
C. <a href="#">Real Numbers</a>	<a href="#">32</a>
D. <a href="#">Ordinals</a>	<a href="#">41</a>
<a href="#">Chapter 5 Normal Form</a>	<a href="#">52</a>
A. <a href="#">Combinatorial Lemma on Semigroups</a>	<a href="#">52</a>
B. <a href="#">The <math>\omega</math> Map</a>	<a href="#">54</a>
C. <a href="#">Normal Form</a>	<a href="#">58</a>
D. <a href="#">Application to Real Closure</a>	<a href="#">73</a>
E. <a href="#">Sign Sequence</a>	<a href="#">76</a>
<a href="#">Chapter 6 Lengths and Subsystems which are Sets</a>	<a href="#">95</a>
<a href="#">Chapter 7 Sums as Subshuffles, Unsolved Problems</a>	<a href="#">104</a>

	<u>page</u>
<u>Chapter 8</u> <u>Number Theory</u>	<u>111</u>
<u>A.</u> <u>Basic Results</u>	<u>111</u>
<u>B.</u> <u>Partial Results and Unsolved Problems</u>	<u>114</u>
<u>Chapter 9</u> <u>Generalized Epsilon Numbers</u>	<u>121</u>
<u>A.</u> <u>Epsilon Numbers with Arbitrary Index</u>	<u>121</u>
<u>B.</u> <u>Higher Order Fixed Points</u>	<u>124</u>
<u>C.</u> <u>Sign Sequences for Fixed Points</u>	<u>129</u>
<u>D.</u> <u>Quasi <math>\kappa</math> type Numbers</u>	<u>135</u>
<u>E.</u> <u>Sign Sequences in Quasi Case</u>	<u>138</u>
<u>Chapter 10</u> <u>Exponentiation</u>	<u>143</u>
<u>A.</u> <u>General Theory</u>	<u>143</u>
<u>B.</u> <u>Specialization to Purely Infinite Numbers</u>	<u>156</u>
<u>C.</u> <u>Reduction to the Function <math>g</math></u>	<u>167</u>
<u>D.</u> <u>Properties of <math>g</math> and Explicit Results</u>	<u>175</u>
<u>References</u>	<u>191</u>
<u>Index</u>	<u>192</u>

## PREFACE

The aim of this book is to give a systematic introduction to the theory of surreal numbers based on foundations that are familiar to most mathematicians. I feel that the surreal numbers form an exciting system which deserves to be better known and that therefore an exposition like this one is needed at present. The subject is in such a pioneering state that it appears that there are many results just on the verge of being discovered and even concepts that still are waiting to be defined.

One might claim that one should wait till the theory of surreal numbers is more fully established before publishing a book on this subject. Such a comment reminds me of the classic joke about the person who is afraid of drowning and has vowed never to step into water until he has learned how to swim. In fact, the time is ripe for such a book and furthermore the book itself should contribute to developing the subject with the help of creative readers.

The subject has suffered so far from isolation with pockets of people in scattered parts of the world working on those facets of the subject that interest them. I hope that this book will play a role in eliminating this isolation and bringing together the mathematicians interested in surreal numbers.

The book is thus a reflection of my own personal interest. For example, Martin Kruskal has developed the theory of exponentiation from a somewhat different point of view and carried it in different directions from the presentation in this book. Also, I recently received correspondence from Norman Alling who has recently done work on a facet of the theory of surreal numbers not discussed in the book. With greater communication all this and more could play a role in a future edition.

The basic material is found in chapters 2 through 5. The later chapters are more original and more specialized. Although room for

future improvement exists everywhere, chapters 7 and 8 are in an especially pioneering position: this is where the greatest opportunity seems to exist for knowledgeable readers to obtain new results.

#### ACKNOWLEDGEMENTS

I would like to thank the following people for their help in connection with the manuscript.

First, there is Professor Larry Corwin who took time from his very busy schedule to do a great deal of proofreading. He is responsible for many improvements in the exposition throughout the manuscript. On the other hand, I take full responsibility for any faults in the exposition which still remain. Professor Joe D'Atri and Jim Maloney, a graduate student, have also helped with some proofreading. Also, I should mention Professor Barbara Osofsky who much earlier had read a preliminary draft of chapters one and two and made many valuable suggestions.

Finally, I mention the contribution of two secretaries of the Rutgers Mathematics Department. Mary Anne Jablonski, the co-ordinator, took care of numerous technical details, and Adelaide Boulle did an excellent job of turning my handwritten draft into typescript.

LONDON MATHEMATICAL SOCIETY  
LECTURE NOTE SERIES

Edited by PROFESSOR J. W. S. CASSELS  
*Department of Pure Mathematics and Mathematical Statistics  
16 Mill Lane, Cambridge, CB2 1SB, England*

with the assistance of  
G. R. Allan (Cambridge)  
P. M. Cohn (London)  
F. E. Browder (Chicago)  
M. W. Hirsch (Berkeley)  
G.-C. Rota (M.I.T.)

An Introduction to the Theory of  
**Surreal Numbers**

HARRY GONSHOR  
*Mathematics Department, Rutgers University*

The **surreal numbers** form a system which includes both ordinary real numbers and the ordinals. Since their introduction by J. H. Conway, the theory of **surreal numbers** has seen a rapid development revealing many natural and exciting properties. These notes provide a formal introduction to the theory. The topics covered include exponentiation and generalised  $\omega$ -numbers.

Professor Gonshor has presented here the basic results and key ideas in a clear and lucid style. The subject is still new and he is able to lead the reader through to some of the outstanding problems in the field.

The prerequisites are minimal, so undergraduates, research students and professionals in mathematics will find this a readily-accessible introduction.

ISBN 0-521-31205-1



9 780521 312051