Metodologias Multicritérios de Apoio à Decisão

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General overview of the MACBETH approach

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INTRODUCTION

MACBETH is an interactive approach for cardinal measurement of judgements about the degrees to which the elements of a finite set A possess a property P. The name *MACBETH, Measuring Attractiveness by a Categorical Based Evaluation Technique*, comes from the fact that we conceived our approach with the aim of facilitating the measurement of (degrees of) attractiveness in decision processes. Nevertheless, MACBETH can also be applied to measure other properties in domains of knowledge others than Decision Sciences, such as in Psychophysics or in Social Sciences.

In this article we offer a general overview of MACBETH where we introduce some modifications improving the initial technical formulation of [Bana e Costa and Vansnick, 1993 and 1994a], although the basic conceptual ideas remain the same.

Objectives and questioning mode of MACBETH

Let A be a finite set of potential actions for which an actor D wants our aid in quantifying his judgements of attractiveness of the elements of A. In such a decision context, the MACBETH approach facilitates the construction of a numerical scale $v: A \rightarrow \Re: a \rightarrow v(a)$ such that:

- Not only, $v(a)$ numerically represents the attractiveness of action a for D in the (substantive) sense that:

(1) $\forall a,b \in A$, $v(a) > v(b)$ if and only if D judges a more attractive than b $(a\,P\,b)$

- but also, the positive difference $v(a) - v(b)$ numerically represents the difference of attractiveness between the actions a and b for D in the sense that:

(2) $\forall a,b,c,d \in A$ with $a\,P\,b$ and $c\,P\,d$, $[v(a)-v(b)]/[v(c)-v(d)]$ reflects the ratio (that D feels with greater or lesser precision) of the differences of attractiveness between a and b on the one hand and c and d on the other hand.

Obviously, such a scale v satisfies the conditions defining the notion of "measurable value function" (cf. [Dyer and Sarin, 1979]), for from (2) immediately follows that $\forall a,b,c,d \in A$ with $a\,P\,b$ and $c\,P\,d$, $v(a)-v(b) > v(c)-v(d)$ if and only if the difference of attractiveness between a and b is larger than the difference of attractiveness between c and d.

Several techniques have been suggested in the literature for the construction of such a value function, but the difficulty of the questions that D is asked to answer makes that notion hardly operational in practice. One of the key advantages of MACBETH is precisely that its questioning procedure is particularly intelligible and straightforward, for it involves only two actions in each question. For this purpose, MACBETH makes use of the *notion of difference of attractiveness* between two actions of A.

In fact, $\forall\ a, b \in A$ with $a\,Pb$, MACBETH's questioning mode consists of asking D to express an *absolute judgement* of difference of attractiveness by assigning the pair (a, b) to one of three semantic categories:

C_2 → ***weak** difference of attractiveness*
C_4 → ***strong** difference of attractiveness*
C_6 → ***extreme** difference of attractiveness*

or, in the case of hesitation, to one of three intermediate categories:

C_1 → ***very weak** difference of attractiveness (from nil to weak)*
C_3 → ***moderate** difference of attractiveness (from weak to strong)*
C_5 → ***very strong** difference of attractiveness (from strong to extreme)*.

From a mathematical point of view, the categories C_1, C_2, C_3, C_4, C_5 and C_6 are asymmetric binary relations which constitute a partition of $P = \{(a,b) \in A{\times}A \mid D$ judges a more attractive than $b\}$.

The notion of absolute judgement has also been used by Saaty [1977, 1980] in the questioning mode of his Analytical Hierarchy Process (AHP). In spite of this common basic idea, there is significant fundamental differences between MACBETH and AHP. First, in MACBETH the absolute judgements concern *differences* (of attractiveness) whereas in Saaty's method they concern *ratios* (of priority, or of importance). Moreover, and this a key distinction between the technical procedures behind the two approaches, for determining a numerical scale on A, Saaty *a priori* associates a single arbitrarily fixed number to each of his categories of ratios, whereas in MACBETH:

1. we associate to each of our six categories C_1 to C_6 an <u>interval</u> of the real line, as shown in figure 1, and
2. the thresholds s_i (i = 1, 2, 3, 4, 5, 6) defining the intervals associated with our categories <u>are not</u> *a priori* fixed; on the contrary, they are determined simultaneously with the numerical scale v that we are looking for.

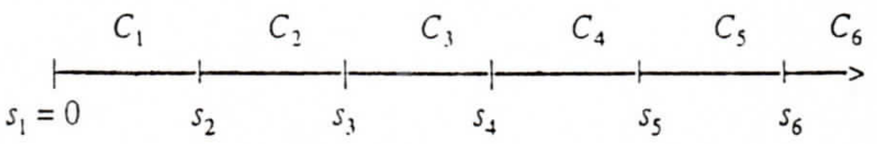

Figure 1 – Representation of MACBETH's categories in the real line

More accurately, once D has assigned each pair $(a, b) \in P$ to one of the six semantic categories of difference of attractiveness, MACBETH tries to determine simultaneously:

- an *application* $v: A \to \mathfrak{R}$ associating each element a of A with a real number $v(a)$

and

- six real numbers s_1, s_2, s_3, s_4, s_5 and s_6 such that

(3) $$0 = s_1 < s_2 < s_3 < s_4 < s_5 < s_6$$

(4) $\forall a, b \in A$:

$$\begin{cases} \forall k \in \{1, 2, 3, 4, 5\}: & s_k < v(a) - v(b) < s_{k+1} \text{ if and only if } (a, b) \in C_k \\ \text{and} & s_6 < v(a) - v(b) \text{ if and only if } (a, b) \in C_6. \end{cases}$$

Let us see how MACBETH interacts with D to reach this purpose, assuming, for reasons of clarity of exposition, that the matter here is only the cardinal component of the problem. That is to say that we will assume the following hypotheses:

Hypothesis H1 The binary relation P modelling the ordinal judgements of attractiveness of D about the elements of A, is asymmetric, negatively transitive, and connected (that is, P is a total strict order).

Hypothesis H2 $A = \{a_1, a_2, \ldots, a_{n-1}, a_n\}$ such that

$$\forall i \neq j \in \{1, 2, \ldots, n\}, a_i \, P \, a_j \text{ if and only if } i > j$$

that is, the complete ranking of the elements of A by order of decreasing attractiveness is $a_n \, P \, a_{n-1} \, P \ldots P \, a_2 \, P \, a_1$.

In these conditions, to collect D's absolute judgements of difference of attractiveness between the elements of A it is sufficient to fill in, row by row, the upper part of the $n \times n$ matrix

	a_n	a_{n-1}	.	.	.	a_2	a_1
a_n		$a_{n,n-1}$	.	.	.	$a_{n,2}$	$a_{n,1}$
a_{n-1}			.	.	.	$a_{n-1,2}$	$a_{n-1,1}$
.				.	.	.	.
.					.	.	.
.						.	.
a_2							$a_{2,1}$
a_1							

where, $\forall i > j \in \{1, 2, \ldots, n\}$ and $\forall k \in \{1, 2, 3, 4, 5, 6\}$:

$$a_{i,j} = k \text{ if and only if } (a_i, a_j) \in C_k.$$

Cardinal consistency

In theory, the search for an application $v: A \rightarrow \Re$ satisfying (4) is a particular case of the problem of the *constant threshold representation* of a *m*-tuple of binary relations, studied by Doignon [1987] who stated a necessary and sufficient condition (cyclone condition) for the existence of a solution.

Bana e Costa and Vansnick [1994b] prove that, when hypothesis H1 is satisfied (*P* is a total strict order), Doignon's condition is a necessary and sufficient one for the existence of an application $v: A \rightarrow \Re$ satisfying (4) and (3). In practice, it is unfortunately very difficult to test the theoretical condition of Doignon. It is the reason why we developed another path for verifying the existence of a solution for our problem. It consists in solving the following linear program (called *Mc*1), the variables of which are $v(a)\ (a \in A)$, $s_1, s_2, s_3, s_4, s_5, s_6$ and c:

Min c

s.t.

r0) all variables ≥ 0

r1) $s_1 = 0$

r2) $v(a_1) = 0$ (see hypothesis H2: $\forall a \in A,\ aPa_1$)

r3) $\forall k \in \{2, 3, 4, 5, 6\}$: $s_k - s_{k-1} \geq 1000$

r4) $\forall k \in \{1, 2, 3, 4, 5, 6\}$, $\forall (a,b) \in C_k$: $v(a) - v(b) \geq s_k + 1 - c$

r5) $\forall k \in \{1, 2, 3, 4, 5\}$, $\forall (a,b) \in C_k$: $v(a) - v(b) \leq s_{k+1} - 1 + c$

Bana e Costa and Vansnick [1994b] prove that an application $v: A \rightarrow \Re$ satisfying (3) and (4) exists if and only if the optimal solution c_{min} of *Mc*1 is equal to 0. When $c_{min} \neq 0$ there exists some cardinal inconsistency in the absolute judgements of *D*, in the sense that the desired numerical representation is not possible. In this case, it is important to try to identify the judgements that are causing trouble in order to discuss with *D*. Two linear programs, called *Mc*3 and *Mc*4, have been developed for that purpose.

Some new non negative variables intervene in *Mc*3, together with $v(a)$ $(a \in A)$, s_1, s_2, s_3, s_4, s_5 and s_6: $\alpha(a,b)$ and $\delta(a,b)$ $[(a,b) \in C_1 \cup C_2 \cup ... \cup C_6]$ and $\beta(a,b)$ and $\gamma(a,b)$ $[(a,b) \in C_1 \cup C_2 \cup ... \cup C_5]$. These variables are defined by:

r6) $\forall k \in \{1, 2, 3, 4, 5, 6\}$ and $\forall (a,b) \in C_k$: $v(a) - v(b) = s_k + 1 - \alpha(a,b) + \delta(a,b)$,

r7) $\forall k \in \{1, 2, 3, 4, 5\}$ and $\forall (a,b) \in C_k$: $v(a) - v(b) = s_{k+1} - 1 + \beta(a,b) - \gamma(a,b)$.

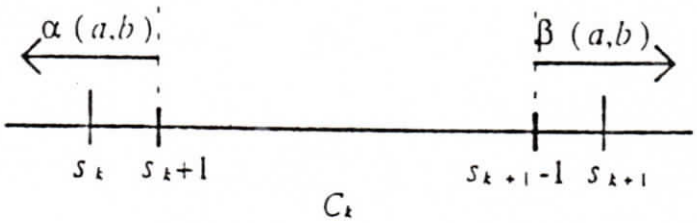

Figure 2: Variables $\alpha(a,b)$ and $\beta(a,b)$ for k < 6

The variables $\alpha(a,b)$ and $\beta(a,b)$ (see figure 2) are particularly interesting for they enable the immediate identification of the ordered pairs (a,b) that can be the cause of inconsistency. The objective function is the same in *Mc*3 and *Mc*4:

$$\text{Min}\left[\sum_{(a,b)\in C_1\cup C_2\cup\ldots\cup C_6} \alpha(a,b) + \sum_{(a,b)\in C_1\cup C_2\cup\ldots\cup C_5} \beta(a,b)\right]$$

and the constraints are:

- for *Mc*3: *r0*, *r1*, *r2*, *r3*, *r4* and *r5* (both with *c* replaced by c_{min}), *r6* and *r7*;
- for *Mc*4: *r0*, *r1*, *r2*, *r3*, *r6* and *r7*.

Note that the two programs only differ on the constraints *r4* and *r5* which are not included in *Mc*4. The aim of *Mc*3 is to detect all the possible sources of inconsistency; in particular, it can be useful to identify the cyclones related to the theoretical condition of Doignon (see Bana e Costa and Vansnick [1994b]). *Mc*4 suggests a restricted number of modifications of category to tentatively reach cardinal consistency. Let us mention that these suggestions must be taken simply as a basis for discussion, *D* being free to modify his initial responses in whatever way he wishes. MACBETH belongs to the (up to now) restricted class of interactive approaches that follow a *learning paradigm* in decision aiding, allowing *D* to feel free in revising his judgements and, if he wants to go back anytime in his process of judging. If in the course of his cognitive process *D* decides to revise some of his initial judgements, making some modifications of category, the consistency of the new matrix of revised judgements will be tested again, thus re-starting the MACBETH interactive cycle of elaboration, modification and or validation of *D* absolute judgements of difference of attractiveness (see figure 4).

Let us add that, before applying program *Mc*1 to the judgements of *D*, it is very interesting, from a learning perspective, to start by testing the following condition (which follows immediately from (4)) called **semantic consistency condition**:

$$\forall a,b,c \in A \text{ with } aPb \text{ and } bPc \text{ and } \forall k,k' \in \{1,2,3,4,5,6\}, \text{ if } (a,b) \in C_k \text{ and } (b,c) \in C_{k'}, \text{ one must have } (a,c) \in C_{k''} \text{ with } k'' \geq \max\{k,k'\}.$$

This condition (which is implied by the cyclone condition) can easily be tested by verifying that:

In each row of D's matrix of judgements of difference of attractiveness the values $a_{i,j}$ do not decrease from left to right, and in each of its columns the values $a_{i,j}$ do not increase from top to bottom.

This is quite an easy practical test, which immediately shows up the judgements that can cause trouble when semantic consistency is not verified, thus creating learning conditions for launching an interactive discussion with *D*, where consistency can be reached by appropriately changing one or several of his initial judgements. This test is perfectly inserted in the constructive principles of our approach.

MACBETH's suggestion of a value scale: the program *Mc2*

As stated above, when c_{min} equals 0, one can be sure that there are real numbers $v(a)$ $(a \in A)$, s_1, s_2, s_3, s_4, s_5 and s_6 verifying (3) and (4). The problem now is how to determine such numbers. Note that, when semantic consistency is verified, it can be interesting to give directly a suggestion of a numerical scale even when $c_{min} \neq 0$. Specially in multi-party or time-constrained decision processes, this can be a more pragmatic path for measurement than the more time-consuming revision of theoretically inconsistent judgements based upon the suggestions of (*Mc*3 and) *Mc*4. To give an answer to these two points, the MACBETH approach includes another linear program, called ***Mc2***, the variables of which are $v(a)$ $(a \in A)$, s_1, s_2, s_3, s_4, s_5 and s_6: $\alpha(a,b)$ and $\delta(a,b)$ $[(a,b) \in C_6]$ and $\varepsilon(a,b)$ and $\eta(a,b)$ $[(a,b) \in C_1 \cup C_2 \cup \ldots \cup C_5]$

$$Min \;\left\{ \sum_{\substack{(a,b) \in C_k,\\ k \in \{1,2,3,4,5\}}} [\varepsilon(a,b) + \eta(a,b)] \; + \sum_{(a,b) \in C_6} \alpha(a,b) \right\}$$

s.t.

r0, *r1*, *r2*, *r3*, *r4* and *r5* (both with c replaced by c_{min})

r8) $\quad \forall (a,b) \in C_6 : v(a) - v(b) = s_6 + 1 - \alpha(a,b) + \delta(a,b)$

r9) $\quad \forall k \in \{1, 2, 3, 4, 5\}, \; \forall (a,b) \in C_k :$

$$v(a) - v(b) = (s_k + s_{k+1})/2 + \varepsilon(a,b) - \eta(a,b)$$

(see figure 3).

The formulation given to the objective-function of *Mc*2 results from our wish to arrive at numerical values for the difference of attractiveness of pairs assigned to the same category such that, if possible, these pairs are closer among themselves than pairs assigned to different categories. Note also that $\varepsilon(a,b).\eta(a,b) = 0$ in any basic solution of *Mc*2.

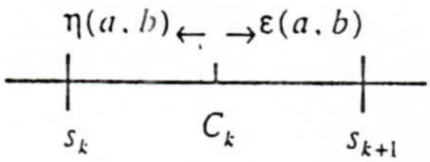

Figure 3 – Variables $\eta(a,b)$ and $\varepsilon(a,b)$

The scheme in figure 4 offers a graphic overview of the cyclic interactive process of the MACBETH approach. As figure 4 suggests, once a numerical scale is obtained (by the optimal solution of *Mc*2) the interaction with *D* can go on to the discussion of the cardinality of the scale, that is, to the validation of condition (2) (note that condition (1) is certainly satisfied). Probably, the most friendly way to ease this discussion will be to plot the numbers $v(a)$ $(a \in A)$ on an oriented axis, and than verify with *D* if the relative distances between points (actions) correspond to *D*'s relative differences of attractiveness. If so, the application $v: A \rightarrow \Re: a \rightarrow v(a)$ can be finally taken as an interval scale quantifying the attractiveness of the actions of *A* for *D*.

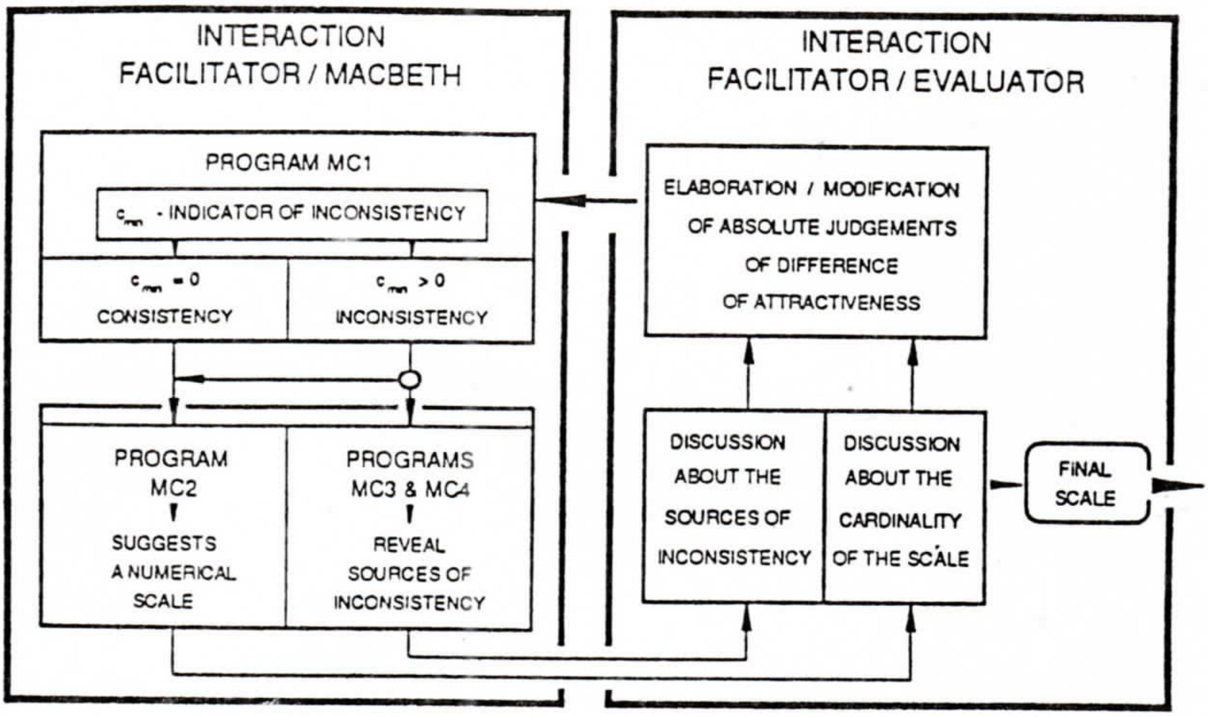

Figure 4 – Interactive scheme of MACBETH

Final comments

In Measurement Theory terminology, MACBETH is an interactive approach for mapping into a real scale various degrees of a property of the elements of a finite set *A*. The originality of MACBETH's questioning procedure is the possibility of establishing a constructive path towards cardinal measurement in both quantitative and substantive meaningful terms, avoiding the operational problems recognised as a weakness of other procedures. The use of the notion of semantic absolute judgements pays a key role here, and the simplicity, interactivity and constructiveness of our approach inserts it in the modern paradigms of decision aid.

In practice, the use of MACBETH strongly facilitates the interaction with the evaluators. As a matter of fact, in all the real-world decision aid cases where we have been applying our approach (see, for example, [Bana e Costa and Vansnick, 1995]) the evaluators not only felt comfortable in answering MACBETH's questions, emphasising the simplicity of the questioning procedure and the usefulness of MACBETH's suggestions to elaborate or revise judgements, but also they were very confident in the results of the interactive learning processes.

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