Problem Set 1 Solutions

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1.1.1

We have

$$\nabla f(x,y) = \begin{pmatrix} 2x + \beta y + 1 \\ 2y + \beta x + 2 \end{pmatrix}$$

Setting $\nabla f(x,y) = 0$, we obtain the system of equations

$$\begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This system has a unique solution (a unique stationary point) except when

$$\beta^2 = 4.$$

If $\beta^2 = 4$, it can be verified that there is no solution to the above system (no stationary point). Assuming $\beta^2 \neq 4$, for the stationary point to be a local minimum, the Hessian matrix of f, which is

$$Q = \begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix},$$

must be positive semidefinite. But if this is so, $f(x,y)$ will be a convex quadratic function and each local minimum will be global.

The Hessian Q will be positive definite if and only if $\beta^2 < 4$ and positive semidefinite if $\beta^2 = 4$, in which case there is no stationary point by the preceding discussion.

Thus, if $\beta^2 < 4$, there is a unique stationary point which is a global minimum. If $\beta^2 = 4$, there is no stationary point. If $\beta^2 > 4$, there is a unique stationary point which, however, is not a local minimum.

1.1.2 (b)

We have

$$\nabla f(x,y) = \begin{pmatrix} x + \cos y \\ -x \sin y \end{pmatrix} \qquad \nabla^2 f(x,y) = \begin{pmatrix} 1 & -\sin y \\ -\sin y & -x \cos y \end{pmatrix}$$

Thus the stationary points of f are:

$$\{((-1)^{k+1}, k\pi) \mid k = \text{ integer}\}, \qquad \{(0, k\pi + \pi/2) \mid k = \text{ integer}\}.$$

Of these, the local minima are

$$\{((-1)^{(k+1)}, k\pi) \mid k = \text{ integer}\}.$$

1.1.3

(a) Since the function $f(x^* + \alpha d)$ is minimized at $\alpha = 0$ for all $d \in \Re^n$, we have for all α and i

$$f(x^* + \alpha e_i) \geq f(x^*),$$

which implies that

$$\lim_{\alpha \to 0^+} \frac{f(x^* + \alpha e_i) - f(x^*)}{\alpha} \geq 0, \qquad \lim_{\alpha \to 0^-} \frac{f(x^* + \alpha e_i) - f(x^*)}{\alpha} \leq 0,$$

or

$$\left(\frac{\partial f(x^*)}{\partial x_i}\right) = 0, \qquad \forall\, i.$$

(b) Consider the function $f(y, z) = (z - py^2)(z - qy^2)$, where $0 < p < q$ and let $x^* = (0, 0)$.

We first show that $g(\alpha)$ is minimized at $\alpha = 0$ for all $d \in \Re^2$. We have

$$g(\alpha) = f(x^* + \alpha d) = f(\alpha d) = (\alpha d_2 - p\alpha^2 d_1^2)(\alpha d_2 - q\alpha^2 d_1^2) = \alpha^2 (d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2).$$

Also,

$$g'(\alpha) = 2\alpha(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(-pd_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(d_2 - p\alpha d_1^2)(-qd_1^2).$$

Thus $g'(0) = 0$. Furthermore,

$$\begin{aligned} g''(\alpha) = 2(d_2 - p\alpha d_1^2)(d_2 - q\alpha d_1^2) + 2\alpha(-pd_1^2)(d_2 - q\alpha d_1^2) + 2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) \\ +2\alpha(-pd_1^2)(d_2 - q\alpha d_1^2) + \alpha^2(-pd_1^2)(-qd_1^2) \\ +2\alpha(d_2 - p\alpha d_1^2)(-qd_1^2) + \alpha^2(-pd_1^2)(-qd_1^2). \end{aligned}$$

Thus $g''(0) = 2d_2^2$, which is greater than 0 if $d_2 \neq 0$. If $d_2 = 0$, $g(\alpha) = pq\alpha^4 d_1^4$, which is clearly minimized at $\alpha = 0$.

Therefore, $(0, 0)$ is a local minimum of f along every line that passes through $(0, 0)$.

Let's now show that if $p < m < q$, $f(y, my^2) < 0$ if $y \neq 0$ and that $f(y, my^2) \geq 0$ otherwise. Consider a point of the form (y, my^2). We have $f(y, my^2) = y^4(m - p)(m - q)$. Clearly, $f(y, my^2) < 0$ if and only if $p < m < q$ and $y \neq 0$. In any $\epsilon-$neighborhood of $(0, 0)$, there exists a $y \neq 0$ such that for some $m \in (p, q)$, (y, my^2) also belongs to the neighborhood. Since $f(0, 0) = 0$, we see that $(0, 0)$ is not a local minimum.