Practical Algorithms for Solving the Quartic Equation

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Each of the five algorithms presented here for solving the quartic equation provides:

- stable analytic solutions for any combination of real coefficients,
- formulas that convert easily to code, and
- calculations that use real numbers only.

Each algorithm is a new modification of an existing method that lacks one of these three properties. Ferrari's method[1, pp 237-253] in its common algorithmic version[2, pp 176-177], [3, §4.4.2], [4] and Descartes' method[5, pp 180-187] can become computationally unstable. The National Bureau of Standards (NBS) method[6, pp 17-18] is unnecessarily complicated. The method of Euler[7, pp 256-262] and that of Van der Waerden[8, pp 190-192] and the Digital Library of Mathematical Functions (DLMF)[9, §1.11(iii)] use calculations with complex numbers.

The algorithm inputs are four real coefficients A_3, A_2, A_1, and A_0, and the outputs are the four values Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)\,(Z-Z_2)\,(Z-Z_3)\,(Z-Z_4) \quad \text{for all Z.}$$

The outputs are thus the four solutions of the general quartic equation

$$Z_n^4 + A_3\,Z_n^3 + A_2\,Z_n^2 + A_1\,Z_n + A_0 = 0, \qquad n = 1, 2, 3, 4. \qquad (1)$$

Except for the NBS method, the algorithms begin by calculating $C = A_3/4$, b_2, b_1, and b_0. The last three of these values are coefficients of the equivalent *depressed quartic equation* with no cubic term:

$$T_n^4 + b_2T_n^2 + b_1T_n + b_0 = 0 \qquad n = 1, 2, 3, 4. \qquad (2)$$

The solutions Z_n of (1) are related to the solutions T_n of (2) by $Z_n = T_n - C$. The coefficients b_2, b_2, and b_0 are calculated from C, A_2, A_1, and A_0 as shown in the algorithm tables below.

Part I of this document presents the five algorithms in both their original and modified forms. Notes explain the computational shortcoming of each original algorithm and the fix. Part I concludes with check equations to validate the set of four calculated solutions Z_n. Part II assesses the suitability of each algorithm for general calculation and demonstrates that all of the algorithms are mathematically equivalent to each other. Part III derives the algorithms.

Unless noted otherwise, the radical sign √ denotes the principal square root. The principal square root of a positive real number is the positive square root. The principal square root of a negative real number is the positive imaginary square root. If z is complex with modulus r and argument ϕ such that $-\pi < \phi \le \pi$, then $z = re^{i\phi}$ and the principal square root of z is $\sqrt{z} = \sqrt{r}\, e^{i\phi/2}$.

Analytic methods for solving quartic equations, including the algorithms presented here, require the solution(s) of a corresponding *resolvent cubic equation*. See the companion paper on algorithms for solving cubic equations.

PART I -- The Algorithms and Check Equations

Ferrari's Method

Common Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$,
$b_1 = A_1 - 2A_2C + 8C^3$,
$b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Solve this resolvent cubic equation for real m:

$$m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0.$$

Use a real solution $m > 0$ if it exists. Otherwise, $m = 0$.

Case: $m > 0$

$$Z_{1,2} = \sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 - b_1/\left(2\sqrt{2m}\right)}$$

$$Z_{3,4} = -\sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 + b_1/\left(2\sqrt{2m}\right)}$$

Case: $m = 0$

$$Z_{1,2} = -C \pm \sqrt{-b_2/2 + \sqrt{b_2^2/4 - b_0}}$$

$$Z_{3,4} = -C \pm \sqrt{-b_2/2 - \sqrt{b_2^2/4 - b_0}}$$

where $b_2^2/4 - b_0 \geq 0$ provided that no real $m > 0$ exists.

Modified Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$,
$b_1 = A_1 - 2A_2C + 8C^3$,
$b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Solve this resolvent cubic equation for real m:

$$m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0.$$

Use a real solution $m > 0$ if it exists. Otherwise, $m = 0$.

$$\Sigma = \begin{cases} 1 & \text{if } b_1 > 0 \\ -1 & \text{otherwise} \end{cases}$$

$$R = \Sigma \sqrt{m^2 + b_2 m + b_2^2/4 - b_0}$$

$$Z_{1,2} = \sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 - R}$$

$$Z_{3,4} = -\sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 + R}$$

where the radicand in the formula for R is nonnegative provided that real $m > 0$ is used if it exists.

In his 1545 book *Ars Magna*, Girolamo Cardano provides the earliest known description of Ferrari's method.[1, pp 237-253] Modern algebraic notation had not been invented at the time. To demonstrate the method, Cardano gave rules for solving the depressed quartic equation and then worked out sample problems.

Similar to Cardano's Problem V, the common algorithm [2, pp 176-177], [3, §4.4.2], [4] uses the solution m of the resolvent cubic equation in a divisor. The quotient $b_1/\left(2\sqrt{2m}\right)$ in the formulas for Z_n causes the common algorithm to become computationally unstable as m approaches zero. The calculated m value typically contains a small round-off error not found in b_1. As b_1 and true m approach zero, the round-off error dominates the calculated m value, and the algorithm becomes unstable. The appendix demonstrates a particularly severe case in which the calculated value of solution Z_1 suffers large error even when m is several orders of magnitude greater than the round-off error.

The modified algorithm, a modern generalization of Cardano's Problem VIII, avoids the instability by replacing $b_1/\left(2\sqrt{2m}\right)$ with R. To check the validity of this replacement, add $b_1^2/8$ to both sides of the resolvent cubic equation, divide through by m, and take the square root.

Descartes' Method

Original Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3 / 4$, $b_2 = A_2 - 6C^2$, $b_1 = A_1 - 2A_2C + 8C^3$, $b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Solve this resolvent cubic equation for real y^2:

$$y^6 + 2b_2 y^4 + (b_2^2 - 4b_0)y^2 - b_1^2 = 0.$$

Use a real solution $y^2 > 0$ if it exists. Otherwise, $y^2 = 0$. Value y is the nonnegative square root of y^2.

Case: $y^2 > 0$

$$Z_{1,2} = y/2 - C \pm \sqrt{-y^2/4 - b_2/2 - b_1/(2y)}$$

$$Z_{3,4} = -y/2 - C \pm \sqrt{-y^2/4 - b_2/2 + b_1/(2y)}$$

Case: $y^2 = 0$

$$Z_{1,2} = -C \pm \sqrt{-b_2/2 + \sqrt{b_2^2/4 - b_0}}$$

$$Z_{3,4} = -C \pm \sqrt{-b_2/2 - \sqrt{b_2^2/4 - b_0}}$$

where $b_2^2/4 - b_0 \geq 0$ provided no real $y^2 > 0$ exists.

Modified Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3 / 4$, $b_2 = A_2 - 6C^2$, $b_1 = A_1 - 2A_2C + 8C^3$, $b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Solve this resolvent cubic equation for real y^2:

$$y^6 + 2b_2 y^4 + (b_2^2 - 4b_0)y^2 - b_1^2 = 0.$$

Use a real solution $y^2 > 0$ if it exists. Otherwise, $y^2 = 0$. Value y is the nonnegative square root of y^2.

$$\Sigma = \begin{cases} 1 & \text{if } b_1 > 0 \\ -1 & \text{otherwise} \end{cases}$$

$$R = \Sigma \sqrt{y^4/4 + (b_2/2)y^2 + b_2^2/4 - b_0}$$

$$Z_{1,2} = y/2 - C \pm \sqrt{-y^2/4 - b_2/2 - R}$$

$$Z_{3,4} = -y/2 - C \pm \sqrt{-y^2/4 - b_2/2 + R}$$

where the radicand in the formula for R is nonnegative provided that real $y^2 > 0$ is used if it exits.

The two Descartes algorithms are similar to the corresponding Ferrari algorithms. The Descartes formulas for Z_n become the corresponding Ferrari formulas by substituting $\sqrt{2m}$ for y. Substitute $\sqrt{2m}$ for y in the Descartes resolvent cubic equation and divide through by 8 to obtain the Ferrari resolvent cubic equation.

Like the Ferrari common algorithm, the Descartes original algorithm suffers computational instability as the solution y^2 of the resolvent cubic equation approaches zero. The instability is avoided in the Descartes modified algorithm just as it is in the Ferrari modified algorithm.

NBS Method

Original Algorithm

Given $Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = 0$, find the real root u_1 of the cubic equation

$$u^3 - A_2u^2 + (A_1A_3 - 4A_0)u - (A_1^2 + A_0A_3^2 - 4A_0A_2) = 0$$

and determine the four roots of the quartic as solutions of the two quadratic equations

$$v^2 + \left[\frac{A_3}{2} \mp \left(\frac{A_3^2}{4} + u_1 - A_2\right)^{\frac{1}{2}}\right] v + \frac{u_1}{2} \mp \left[\left(\frac{u_1}{2}\right)^2 - A_0\right]^{\frac{1}{2}} = 0.$$

If all roots of the cubic equation are real, use the value of u_1 which gives real coefficients in the quadratic equation and select signs so that if

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z^2+p_1Z+q_1)(Z^2+p_2Z+q_2)$$

then

$$p_1 + p_2 = A_3, \quad p_1p_2 + q_1 + q_2 = A_2, \quad p_1q_2 + p_2q_1 = A_1, \quad q_1q_2 = A_0.$$

Modified Algorithm

<u>Problem</u>: Given real coefficients A_3, A_2, A_1, and A_0, find Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

<u>Solution</u>: Calculate u_1 as the greatest real solution of the resolvent cubic equation

$$u^3 - A_2u^2 + (A_1A_3 - 4A_0)u + 4A_0A_2 - A_1^2 - A_0A_3^2 = 0.$$

$$\Sigma_g = \begin{cases} 1 & \text{if } A_1 - A_3u_1/2 > 0 \\ -1 & \text{otherwise} \end{cases}$$

$$p_1 = A_3/2 - \sqrt{A_3^2/4 + u_1 - A_2} \qquad p_2 = A_3/2 + \sqrt{A_3^2/4 + u_1 - A_2}$$

$$q_1 = u_1/2 + \Sigma_g \sqrt{u_1^2/4 - A_0} \qquad q_2 = u_1/2 - \Sigma_g \sqrt{u_1^2/4 - A_0}$$

$$Z_{1,2} = -p_1/2 \pm \sqrt{p_1^2/4 - q_1} \qquad Z_{3,4} = -p_2/2 \pm \sqrt{p_2^2/4 - q_2}$$

The NBS original algorithm is unnecessarily complicated and difficult to code. The user is left to perform trial-and-error tests for two of the algorithm steps: 1) if all three solutions of the resolvent cubic equation are real, select u_1 to produce real coefficients in the quadratic equations, and 2) in the two quadratic equations, choose the correct combination of signs to be used in the expressions for the coefficients. The user faces the possibility of performing trial-and-error tests on three cubic-equation solutions and four sign combinations to arrive at the two correct quadratic equations.

The modified algorithm is easy to code and satisfies all requirements of the original algorithm. Choosing u_1 as the greatest real solution of the resolvent cubic equation always provides real coefficients: p_1, p_2, q_1, and q_2. The function Σ_g as defined in the modified algorithm assures that correct signs are selected.

Euler's Method

Original Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$, $b_1 = A_1 - 2A_2C + 8C^3$, $b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Find the three solutions r_1, r_2, and r_3 of the resolvent cubic equation:

$$r_k^3 + (b_2/2)\,r_k^2 + [(b_2^2 - 4b_0)/16]\,r_k - b_1^2/64 = 0.$$

$$T_1 = \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}$$
$$T_2 = \sqrt{r_1} - \sqrt{r_2} - \sqrt{r_3}$$
$$T_3 = -\sqrt{r_1} + \sqrt{r_2} - \sqrt{r_3}$$
$$T_4 = -\sqrt{r_1} - \sqrt{r_2} + \sqrt{r_3}$$

The signs for the $\sqrt{r_k}$ are selected so that

$$\sqrt{r_1}\sqrt{r_2}\sqrt{r_3} = -b_1/8.$$

$$Z_n = T_n - C, \quad n = 1, 2, 3, 4$$

Modified Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$, $b_1 = A_1 - 2A_2C + 8C^3$, $b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Find the three solutions r_1, r_2, and r_3 of the resolvent cubic equation:

$$r_k^3 + (b_2/2)\,r_k^2 + [(b_2^2 - 4b_0)/16]\,r_k - b_1^2/64 = 0.$$

Solution r_1 is the greatest real solution and $r_1 \geq 0$. Solutions $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ are real ($y_2 = y_3 = 0$), or they form a complex conjugate pair ($x_2 = x_3,\ y_2 = -y_3 > 0$).

$\Sigma = 1$ if $b_1 > 0$, $\Sigma = -1$ otherwise.

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{x_2 + x_3 - 2\Sigma\sqrt{x_2x_3 + y_2^2}}$$

$$T_{3,4} = -\sqrt{r_1} \pm \sqrt{x_2 + x_3 + 2\Sigma\sqrt{x_2x_3 + y_2^2}}$$

where $x_2x_3 + y_2^2 \geq 0$.

$$Z_n = T_n - C, \quad n = 1, 2, 3, 4$$

The radical in the Euler original algorithm does not imply the principal square root. Instead the user selects either square root for any two of the $\sqrt{r_k}$. The third $\sqrt{r_k}$ is selected to satisfy $\sqrt{r_1}\sqrt{r_2}\sqrt{r_3} = -b_1/8$. The user only needs to check that the two sides of this equation have the same sign. The resolvent cubic equation guarantees that the two sides have the same absolute value:

$$(r - r_1)(r - r_2)(r - r_3) = r^3 + (b_2/2)r^2 + [(b_2^2 - 4b_0)/16]\,r - b_1^2/64 \text{ for all } r$$

$$\Rightarrow \quad r_1r_2r_3 = b_1^2/64 \quad \Rightarrow \quad \left|\sqrt{r_1}\sqrt{r_2}\sqrt{r_3}\right| = |b_1|/8. \tag{3}$$

The resolvent cubic equation sometimes has two solutions that are a complex conjugate pair. The original algorithm then requires operations on complex numbers, but the modified algorithm does not. In the modified algorithm, all constituents of the T_n formulas are real numbers, and the inner radicand, $x_2x_3 + y_2^2$, is nonnegative.

The modified algorithm's T_n formulas may be expressed more simply as

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{r_2 + r_3 - 2\Sigma\sqrt{r_2r_3}} \qquad T_{3,4} = -\sqrt{r_1} \pm \sqrt{r_2 + r_3 + 2\Sigma\sqrt{r_2r_3}}\,. \tag{4}$$

Solutions $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ of the resolvent cubic equation are real ($y_2 = y_3 = 0$), or they form a complex conjugate pair ($x_2 = x_3,\ y_2 = -y_3 > 0$). In either case, the sum $r_2 + r_3$ equals $x_2 + x_3$, and the product r_2r_3 equals $x_2x_3 + y_2^2$.

Van der Waerden's Method

Original Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$, $b_1 = A_1 - 2A_2C + 8C^3$, $b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Find the three solutions θ_1, θ_2, and θ_3 of the resolvent cubic equation:

$$\theta_k^3 - 2b_2\theta_k^2 + (b_2^2 - 4b_0)\theta_k + b_1^2 = 0.$$

$$T_1 = \tfrac{1}{2}\left[\ \sqrt{-\theta_1} + \sqrt{-\theta_2} + \sqrt{-\theta_3}\ \right]$$

$$T_2 = \tfrac{1}{2}\left[\ \sqrt{-\theta_1} - \sqrt{-\theta_2} - \sqrt{-\theta_3}\ \right]$$

$$T_3 = \tfrac{1}{2}\left[-\sqrt{-\theta_1} + \sqrt{-\theta_2} - \sqrt{-\theta_3}\ \right]$$

$$T_4 = \tfrac{1}{2}\left[-\sqrt{-\theta_1} - \sqrt{-\theta_2} + \sqrt{-\theta_3}\ \right]$$

The signs for the $\sqrt{-\theta_k}$ are selected so that

$$\sqrt{-\theta_1}\,\sqrt{-\theta_2}\,\sqrt{-\theta_3} = -b_1.$$

$$Z_n = T_n - C, \quad n = 1, 2, 3, 4$$

Modified Algorithm

Given: Real coefficients A_3, A_2, A_1, and A_0,

Find: Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

Calculation: $C = A_3/4$, $b_2 = A_2 - 6C^2$, $b_1 = A_1 - 2A_2C + 8C^3$, $b_0 = A_0 - A_1C + A_2C^2 - 3C^4$

Find the three solutions θ_1, θ_2, and θ_3 of the resolvent cubic equation:

$$\theta_k^3 - 2b_2\theta_k^2 + (b_2^2 - 4b_0)\theta_k + b_1^2 = 0.$$

Solution θ_1 is the least real solution and $-\theta_1 \geq 0$. Solutions $\theta_2 = \theta_{x2} + i\theta_{y2}$, and $\theta_3 = \theta_{x3} + i\theta_{y3}$ are real ($\theta_{y2} = \theta_{y3} = 0$), or they form a complex conjugate pair ($\theta_{x2} = \theta_{x3}$, $-\theta_{y2} = \theta_{y3} > 0$).

$$\Sigma = 1 \text{ if } b_1 > 0,\ \Sigma = -1 \text{ otherwise.}$$

$$T_{1,2} = \frac{1}{2}\left[\ \sqrt{-\theta_1} \pm \sqrt{-\theta_{x2} - \theta_{x3} - 2\Sigma\sqrt{\theta_{x2}\theta_{x3} + \theta_{y2}^2}}\ \right]$$

$$T_{3,4} = \frac{1}{2}\left[-\sqrt{-\theta_1} \pm \sqrt{-\theta_{x2} - \theta_{x3} + 2\Sigma\sqrt{\theta_{x2}\theta_{x3} + \theta_{y2}^2}}\ \right]$$

where $\theta_{x2}\theta_{x3} + \theta_{y2}^2 \geq 0$.

$$Z_n = T_n - C, \quad n = 1, 2, 3, 4$$

Van der Waerden derived this method,[8, pp 190-192] and the DLMF presents it in algorithmic form.[9, §1.11(iii)]

The algorithms are similar to the corresponding Euler algorithms. Euler's T_n formulas convert to Van der Waerden's with the following substitutions:

$$r_k = -\theta_k/4, \quad x_k = -\theta_{xk}/4, \quad y_k = -\theta_{yk}/4, \qquad k = 1, 2, 3.$$

Substitute $-\theta_k/4$ for r_k in Euler's resolvent cubic equation and simplify to obtain Van der Waerden's. As with Euler, the Van der Waerden original algorithm requires operations on complex numbers, but the modified algorithm does not.

The T_n formulas in the modified algorithm may be expressed more simply as

$$T_{1,2} = \frac{1}{2}\left[\sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 - 2\Sigma\sqrt{\theta_2\theta_3}}\right] \qquad T_{3,4} = \frac{1}{2}\left[-\sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 + 2\Sigma\sqrt{\theta_2\theta_3}}\right] \quad (5)$$

Solutions $\theta_2 = \theta_{x2} + i\theta_{y2}$ and $\theta_3 = \theta_{x3} + i\theta_{y3}$ of the resolvent cubic equation are real ($\theta_{y2} = \theta_{y3} = 0$), or they form a complex conjugate pair ($\theta_{x2} = \theta_{x3}$, $-\theta_{y2} = \theta_{y3} > 0$). In either case, $-\theta_2 - \theta_3 = -\theta_{x2} - \theta_{x3}$, and $\theta_2\theta_3 = \theta_{x2}\theta_{x3} + \theta_{y2}^2$.

Checking the Solutions

The set of calculated solutions Z_1, Z_2, Z_3 and Z_4 of the quartic equation can be checked against the requirement that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \quad \text{for all } Z.$$

Expand and simplify the right side of this equation, and then equate each coefficient to the corresponding coefficient on the left side to obtain

$$A_3 = -(Z_1+Z_2+Z_3+Z_4)$$

$$A_2 = Z_1Z_2+Z_1Z_3+Z_1Z_4+Z_2Z_3+Z_2Z_4+Z_3Z_4$$

$$A_1 = -(Z_1Z_2Z_3+Z_1Z_2Z_4+Z_1Z_3Z_4+Z_2Z_3Z_4)$$

$$A_0 = Z_1Z_2Z_3Z_4\,.$$

Express each Z_n as the sum of its real and imaginary components: $Z_n = X_n + iY_n$. Solutions Z_1 and Z_2 are either real ($Y_1 = Y_2 = 0$) or they form a complex conjugate pair ($X_1 = X_2$, $Y_1 = -Y_2 > 0$). Solutions Z_3 and Z_4 are either real ($Y_3 = Y_4 = 0$) or they form a complex conjugate pair ($X_3 = X_4$, $Y_3 = -Y_4 > 0$). We now have:

$$A_3 = -(X_1+X_2+X_3+X_4)$$

$$A_2 = X_1X_2+Y_1^2+(X_1+X_2)(X_3+X_4)+X_3X_4+Y_3^2$$

$$A_1 = -[(X_1X_2+Y_1^2)(X_3+X_4) + (X_3X_4+Y_3^2)(X_1+X_2)]$$

$$A_0 = (X_1X_2+Y_1^2)(X_3X_4+Y_3^2).$$

Valid solutions must reproduce the input coefficients according to these check equations.

PART II -- Algorithm Assessment

Algorithm Suitability for General Calculation
Each of the modified algorithms presented here is suitable for general calculation. Each provides:

- stable analytic solutions for any combination of real coefficients,
- formulas that convert easily to code, and
- calculation with real numbers only.

The choice of one of these algorithms over the others is therefore a matter of personal preference.

For introducing students to quartic equations, instructors may prefer Ferrari's method. It is the earliest analytic method, and its derivation relies primarily on the technique of completing the square with which the students should be familiar. Students are thus likely to find its derivation the easiest to understand and remember. Moreover, the same derivation produces both the common algorithm and the modified algorithm.

The NBS modified algorithm has the advantage of not requiring the intermediate depressed quartic equation.

The Euler and Van der Waerden original algorithms are also suitable for users who have the ability to perform operations on complex numbers.

The Ferrari common algorithm and the Descartes original algorithm are not recommended for general calculation because they can become computationally unstable. Even so, they are mathematically useful. In the section below, the Ferrari common algorithm helps demonstrate the mathematical equivalence of the NBS algorithm to the other algorithms presented here. Heikkinen[10] uses the unstable Descartes original algorithm to derive his stable algorithm for calculating a position's geodetic coordinates (longitude, latitude, altitude) given the position's earth-centered, earth-fixed rectangular coordinates on the earth ellipsoid.

The NBS original and modified algorithms are not two different algorithms in the same way that the Ferrari common and modified algorithms are different from each other. The NBS original is not even an algorithm in the sense of a defined sequence of logical and mathematical operations. It is rather a set of detailed requirements. An algorithm which meets all of the requirements can solve the general quartic equation correctly. The modified version is a true algorithm that meets all of the requirements.

Mathematical Equivalence of the Algorithms
Although the algorithms here vary in their suitability for general calculation, they are all mathematically equivalent to each other. This of course must be true because they all produce the correct quartic-equation solutions in theory. This section demonstrates equivalence by showing that any of the algorithms presented can be converted to any of the others. The NBS original algorithm is excluded because it is not a true algorithm, as just pointed out. From this point on, the term "NBS algorithm" refers to the NBS modified algorithm.

The NBS algorithm requires special treatment because it is the only algorithm considered here that solves the general quartic equation directly. We refer to the other algorithms as the depressed algorithms because they solve the equivalent depressed quartic equation and then shift the results to obtain solutions of the general quartic equation. Our demonstration first shows the mathematical equivalence of all of the depressed algorithms and then addresses their equivalence to the NBS algorithm.

The Depressed Algorithm Summary Table below lists the resolvent cubic equations and T_n formulas for all of the depressed algorithms.

Principal-Square-Root Convention for Radicals

The table uses the principal-square-root convention for radicals, so the T_n formulas in the Euler and Van der Waerden original algorithms are recast accordingly. This is accomplished in the Euler original algorithm by replacing $\sqrt{r_3}$ with $-\Sigma s\sqrt{r_3}$ where

$$\Sigma = \begin{cases} 1 & \text{if } b_1 > 0 \\ -1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad s = \begin{cases} 1 & \text{if } \sqrt{r_1}\sqrt{r_2}\sqrt{r_3} \geq 0 \\ -1 & \text{otherwise.} \end{cases} \tag{6}$$

The definitions of these special functions and equation (3) above imply that

$$b_1 = \Sigma|b_1| \qquad \text{and} \qquad \sqrt{r_1}\sqrt{r_2}\sqrt{r_3} = s\left|\sqrt{r_1}\sqrt{r_2}\sqrt{r_3}\right| = s|b_1|/8.$$

The Euler original T_n formulas change

FROM: $T_{1,2} = \sqrt{r_1} \pm \left(\sqrt{r_2} + \sqrt{r_3}\right)$ and $T_{3,4} = -\sqrt{r_1} \pm \left(\sqrt{r_2} - \sqrt{r_3}\right)$

TO: $T_{1,2} = \sqrt{r_1} \pm \left(\sqrt{r_2} - \Sigma s\sqrt{r_3}\right)$ and $T_{3,4} = -\sqrt{r_1} \pm \left(\sqrt{r_2} + \Sigma s\sqrt{r_3}\right)$. (7)

In this revised formulation, the product of terms for each T_n is

$$-\Sigma s\sqrt{r_1}\sqrt{r_2}\sqrt{r_3} = -\Sigma s^2|b_1|/8 = -\Sigma|b_1|/8 = -b_1/8$$

as required by the Euler original algorithm. The Van der Waerden original algorithm is recast in a corresponding fashion.

The function s in (6) accommodates the condition $\sqrt{r_1}\sqrt{r_2}\sqrt{r_3} < 0$, which occurs when one of the r_k, say r_1, is positive real and the other two r_k are negative real:

$$\sqrt{r_2} = i\sqrt{|r_2|},\ \sqrt{r_3} = i\sqrt{|r_3|} \quad \Rightarrow \quad \sqrt{r_2}\sqrt{r_3} = -\sqrt{|r_2|}\sqrt{|r_3|} = -\sqrt{r_2 r_3} < 0.$$

For the Euler and Van der Waerden modified algorithms, the table uses the simplified T_n formulas from (4) and (5).

Greatest Real Solution of the Resolvent Cubic Equations

The algorithm summary table applies the following additional convention. For calculating the T_n, each algorithm except Van der Waerden employs the greatest real solution of its resolvent cubic equation. This solution is m in Ferrari, y^2 in Descartes, and r_1 in Euler. Van der Waerden uses the least real solution θ_1 of its resolvent cubic equation. This convention allows a direct comparison of all the algorithms.

DEPRESSED ALGORITHM SUMMARY TABLE

<table>
<tr><td colspan="3">Given real coefficients b_2, b_1, and b_0, find T_1, T_2, T_3, and T_4 such that
$(T - T_1)(T - T_2)(T - T_3)(T - T_4) = T^4 + b_2T^2 + b_1T + b_0$ for all T.
Definition: $\Sigma = 1$ if $b_1 > 0$, $\Sigma = -1$ otherwise.
Radical $\sqrt{\ }$ denotes the principal square root for all algorithms.</td></tr>
<tr><td rowspan="3">Ferrari's Method</td><td colspan="2">m is greatest real solution of: $m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0$, and $m \ge 0$.</td></tr>
<tr><td>Common Algorithm ($m > 0$) *</td><td>Modified Algorithm</td></tr>
<tr><td>$T_{1,2} = \sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 - b_1/(2\sqrt{2m})}$
$T_{3,4} = -\sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 + b_1/(2\sqrt{2m})}$</td><td>$R = \Sigma\sqrt{m^2 + b_2 m + b_2^2/4 - b_0}$
$T_{1,2} = \sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 - R}$
$T_{3,4} = -\sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 + R}$</td></tr>
<tr><td rowspan="3">Descartes' Method</td><td colspan="2">y^2 is greatest real solution of: $y^6 + 2b_2 y^4 + (b_2^2 - 4b_0)y^2 - b_1^2 = 0$, and $y \ge 0$.</td></tr>
<tr><td>Original Algorithm ($y^2 > 0$) *</td><td>Modified Algorithm</td></tr>
<tr><td>$T_{1,2} = y/2 \pm \sqrt{-y^2/4 - b_2/2 - b_1/(2y)}$
$T_{3,4} = -y/2 \pm \sqrt{-y^2/4 - b_2/2 + b_1/(2y)}$</td><td>$R = \Sigma\sqrt{y^4/4 + (b_2/2)y^2 + b_2^2/4 - b_0}$
$T_{1,2} = y/2 \pm \sqrt{-y^2/4 - b_2/2 - R}$
$T_{3,4} = -y/2 \pm \sqrt{-y^2/4 - b_2/2 + R}$</td></tr>
<tr><td rowspan="3">Euler's Method</td><td colspan="2">r_1 is the greatest real solution of: $r_k^3 + (b_2/2)\, r_k^2 + [(b_2^2 - 4b_0)/16]\, r_k - b_1^2/64 = 0$, and $r_1 \ge 0$.
Solutions $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ are real ($y_2 = y_3 = 0$), or they form a complex conjugate pair ($x_2 = x_3$, $y_2 = -y_3 > 0$).</td></tr>
<tr><td>Original Algorithm</td><td>Modified Algorithm</td></tr>
<tr><td>$T_{1,2} = \sqrt{r_1} \pm \left(\sqrt{r_2} - \Sigma s\sqrt{r_3}\right)$
$T_{3,4} = -\sqrt{r_1} \pm \left(\sqrt{r_2} + \Sigma s\sqrt{r_3}\right)$
$s = 1$ if $\sqrt{r_1}\sqrt{r_2}\sqrt{r_3} \ge 0$, $s = -1$ otherwise.</td><td>$T_{1,2} = \sqrt{r_1} \pm \sqrt{r_2 + r_3 - 2\Sigma\sqrt{r_2 r_3}}$
$T_{3,4} = -\sqrt{r_1} \pm \sqrt{r_2 + r_3 + 2\Sigma\sqrt{r_2 r_3}}$</td></tr>
<tr><td rowspan="3">Van der Waerden Method</td><td colspan="2">θ_1 is the least real solution of: $\theta_k^3 - 2b_2\theta_k^2 + (b_2^2 - 4b_0)\theta_k + b_1^2 = 0$, and $-\theta_1 \ge 0$.
Solutions $\theta_2 = \theta_{x2} + i\theta_{y2}$, and $\theta_3 = \theta_{x3} + i\theta_{y3}$ are real ($\theta_{y2} = \theta_{y3} = 0$), or they form a complex conjugate pair ($\theta_{x2} = \theta_{x3}$, $-\theta_{y2} = \theta_{y3} > 0$).</td></tr>
<tr><td>Original Algorithm</td><td>Modified Algorithm</td></tr>
<tr><td>$T_{1,2} = \frac{1}{2}\left[\sqrt{-\theta_1} \pm \left(\sqrt{-\theta_2} - \Sigma s\sqrt{-\theta_3}\right)\right]$
$T_{3,4} = \frac{1}{2}\left[-\sqrt{-\theta_1} \pm \left(\sqrt{-\theta_2} + \Sigma s\sqrt{-\theta_3}\right)\right]$
$s = 1$ if $\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3} \ge 0$, $s = -1$ otherwise.</td><td>$T_{1,2} = \frac{1}{2}\left[\sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 - 2\Sigma\sqrt{\theta_2\theta_3}}\right]$
$T_{3,4} = \frac{1}{2}\left[-\sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 + 2\Sigma\sqrt{\theta_2\theta_3}}\right]$</td></tr>
<tr><td colspan="3">* T_n formulas for the Ferrari common algorithm and the Descartes original algorithm for the case $m = y^2 = 0$ are
$T_{1,2} = \pm\sqrt{-b_2/2 + \sqrt{b_2^2/4 - b_0}}$ $T_{3,4} = \pm\sqrt{-b_2/2 - \sqrt{b_2^2/4 - b_0}}$.</td></tr>
</table>

Resolvent Cubic Equations in the Depressed Algorithms

In the table above, each resolvent cubic equation converts to every other by applying the transformation

$$2m = y^2 = 4r_k = -\theta_k$$

and simplifying to standard form. The greatest real solutions in Ferrari, Descartes, and Euler and the least real solution in Van der Waerden are related by

$$2m = y^2 = 4r_1 = -\theta_1. \tag{8}$$

The algorithms use these values to calculate the T_n.

The greatest real solution m of the Ferrari resolvent cubic equation is nonnegative, as we now demonstrate. The resolvent cubic equation is

$$m^3 + b_2\,m^2 + (b_2^2/4 - b_0)m - b_1^2/8 \;=\; 0.$$

The constant coefficient, $-b_1^2/8$, is less than or equal to zero. If $b_1 = 0$, then $m = 0$ is a solution. Otherwise, the cubic on the left side of the equation is negative at $m = 0$, but the cubic must eventually increase to zero as m increases to a sufficiently large positive value. This m value is a positive real solution. Thus $m \geq 0$, and by (8) we have

$$2m = y^2 = 4r_1 = -\theta_1 \;\geq\; 0. \tag{9}$$

These values that are used to calculate the T_n are all nonnegative.

T_n Formulas for Ferrari and Descartes Algorithms

Convert the Ferrari common T_n formulas for $m > 0$ to the Ferrari modified T_n formulas by solving the resolvent cubic equation for $b_1^2/(8m)$ and taking the square root.

$$m^3 + b_2\,m^2 + (b_2^2/4 - b_0)m - b_1^2/8 \;=\; 0 \quad\Rightarrow\quad b_1^2/(8m) \;=\; m^2 + b_2\,m + b_2^2/4 - b_0$$

$$\Rightarrow \qquad b_1/\left(2\sqrt{2m}\,\right) \;=\; \Sigma\sqrt{m^2 + b_2 m + b_2^2/4 - b_0} \;=\; R \qquad (m > 0)$$

$$\text{where}\quad \Sigma = 1 \text{ if } b_1 > 0,\ \Sigma = -1 \text{ otherwise}$$

The term $b_1/\left(2\sqrt{2m}\,\right)$ in the common T_n formulas is replaced with R to produce the Ferrari modified T_n formulas.

The case $m = 0$ implies that $b_1 = 0$, $\Sigma = -1$, and $R = -\sqrt{b_2^2/4 - b_0}$. The Ferrari modified algorithm produces the same formula for T_n as the Ferrari common algorithm for $m = 0$. The Ferrari common and modified algorithms are therefore mathematically equivalent to each other for all m.

The transform $m = y^2/2$ from (9) converts the Ferrari common and modified T_n formulas to the corresponding formulas in the Descartes algorithms. Thus, the Ferrari and Descartes algorithms convert to each other and are all equivalent to each other.

T_n Formulas for Euler and Van der Waerden Algorithms

Using the principal-square-root convention for radicals, the T_n formulas for the Euler original algorithm are

$$T_{1,2} = \sqrt{r_1} \pm \left(\sqrt{r_2} - \Sigma s\sqrt{r_3}\right) \qquad T_{3,4} = -\sqrt{r_1} \pm \left(\sqrt{r_2} + \Sigma s\sqrt{r_3}\right) \tag{10}$$

where

$$\Sigma = \begin{cases} 1 & \text{if } b_1 > 0 \\ -1 & \text{otherwise} \end{cases} \quad \text{and} \quad s = \begin{cases} 1 & \text{if } \sqrt{r_1}\sqrt{r_2}\sqrt{r_3} \geq 0 \\ -1 & \text{otherwise} \end{cases}, \tag{11}$$

and where r_1, r_2, and r_3 are solutions of the resolvent cubic equation

$$r_k^3 + (b_2/2)\, r_k^2 + [(b_2^2 - 4b_0)/16]\, r_k - b_1^2/64 = 0.$$

By (9), the greatest real solution r_1 is nonnegative, as is $\sqrt{r_1}$:

$$r_1 \geq 0 \qquad \Rightarrow \qquad \sqrt{r_1} \geq 0. \tag{12}$$

Equation (3) shows that $r_1r_2r_3 = b_1^2/64 \geq 0$. Therefore,

$$r_1 \geq 0 \quad \text{and} \quad r_1r_2r_3 = b_1^2/64 \geq 0 \quad \Rightarrow \quad r_2r_3 \geq 0. \tag{13}$$

The product r_2r_3 is a nonnegative real number. Thus $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ are real ($y_2 = y_3 = 0$), or they form a complex conjugate pair ($x_2 = x_3,\ y_2 = -y_3$). If real, then they cannot have opposite signs. This restriction on r_2 and r_3 implies that each parenthetical expression in (10) is either real or pure imaginary.

Conversion of the T_n formulas in (10) to those in the Euler modified algorithm starts by replacing each parenthetical expression with the radical of its square:

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{r_2 + r_3 - 2\Sigma s\sqrt{r_2}\sqrt{r_3}} \qquad T_{3,4} = -\sqrt{r_1} \pm \sqrt{r_2 + r_3 + 2\Sigma s\sqrt{r_2}\sqrt{r_3}}. \tag{14}$$

T_1 and T_2 in (14) each have the same value as in (10) unless $\sqrt{r_2} - \Sigma s\sqrt{r_3}$ happens to be either negative real or negative imaginary. In that case, T_1 in (10) is T_2 in (14) and T_2 in (10) is T_1 in (14). T_3 and T_4 are correspondingly affected by the value of $\sqrt{r_2} + \Sigma s\sqrt{r_3}$.

As an option to prevent the T_n from flipping values between (10) and (14), use the following convention: select $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ so that $|x_2| \geq |x_3|$ and $y_2 = -y_3 \geq 0$. The convention assures that the parenthetical expressions in (10) are either nonnegative real or nonnegative imaginary. The convention does not affect (14), which is symmetrical with respect to r_2 and r_3.

Equation (12) implies that the formula for s in (11) simplifies to

$$s = \begin{cases} 1 & \text{if } \sqrt{r_2}\sqrt{r_3} \geq 0 \\ -1 & \text{otherwise} \end{cases} \qquad \Rightarrow \qquad \sqrt{r_2}\sqrt{r_3} = s\left|\sqrt{r_2}\sqrt{r_3}\right| = s\left|\sqrt{r_2r_3}\right|.$$

This result and (13) imply that

$$s\sqrt{r_2}\sqrt{r_3} = s^2\left|\sqrt{r_2r_3}\right| = \left|\sqrt{r_2r_3}\right| = \sqrt{r_2r_3}\,.$$

The T_n formulas in (14) convert to the simplified T_n formulas in the Euler modified algorithm.

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{r_2 + r_3 - 2\Sigma\sqrt{r_2r_3}} \qquad T_{3,4} = -\sqrt{r_1} \pm \sqrt{r_2 + r_3 + 2\Sigma\sqrt{r_2r_3}}. \tag{15}$$

The Euler original and modified algorithms are therefore mathematically equivalent.

The transform $r_k = -\theta_k/4$ converts the original and modified Euler T_n formulas to the corresponding formulas in the Van der Waerden algorithms. Thus, the Euler and Van der Waerden algorithms are all equivalent to each other.

Equivalence of T_n Formulas from All Eight Depressed Algorithms

So far, we have the eight depressed algorithms grouped into two sets of equivalent algorithms:

Set 1: Ferrari common and modified, Descartes original and modified
Set 2: Euler original and modified, and Van der Waerden original and modified.

We show that the two sets are equivalent by converting the Euler modified T_n formulas from Set 2 to the Ferrari modified T_n formulas from Set 1. Solutions r_k of the Euler resolvent cubic equation must satisfy the requirement:

$$(r - r_1)\,(r - r_2)\,(r - r_3) \;=\; r^3 + (b_2/2)r^2 + [(b_2^2 - 4b_0)/16]\,r - b_1^2/64 \quad \text{for all r.}$$

Expand and simplify the left side, then equate coefficients with the corresponding coefficients on the right. Results for the quadratic and linear coefficients are:

$$-(r_1 + r_2 + r_3) \;=\; b_2/2 \qquad\qquad r_1r_2 + r_1r_3 + r_2r_3 \;=\; (b_2^2 - 4b_0)/16.$$

Solve these two equations for $r_2 + r_3$ and for r_2r_3 as functions of r_1.

$$r_2 + r_3 \;=\; -r_1 - b_2/2 \qquad\qquad r_2r_3 \;=\; r_1^2 + (b_2/2)r_1 + (b_2^2 - 4b_0)/16$$

Substitute these two expressions into the Euler modified T_n equations and simplify.

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{-r_1 - b_2/2 - \Sigma\sqrt{4r_1^2 + 2b_2r_1 + b_2^2/4 - b_0}}$$

$$T_{3,4} = -\sqrt{r_1} \pm \sqrt{-r_1 - b_2/2 + \Sigma\sqrt{4r_1^2 + 2b_2r_1 + b_2^2/4 - b_0}}$$

Apply the transform $r_1 = m/2$ from (9) to produce the Ferrari modified T_n formulas:

$$T_{1,2} = \sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 - R} \qquad\qquad T_{3,4} = -\sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 + R}$$

where $$R = \Sigma\sqrt{m^2 + b_2m + b_2^2/4 - b_0}\,.$$

The Euler modified algorithm from Set 2 converts to the Ferrari modified algorithm from Set 1. Thus, all eight depressed algorithms comprising the two sets are mathematically equivalent to each other.

Equivalence of the NBS Algorithm to the Depressed Algorithms

The eight depressed algorithms are also mathematically equivalent to the NBS algorithm as now demonstrated by constructing the NBS algorithm from the Ferrari algorithms.

The NBS algorithm is related to the Ferrari algorithms through u_1, the greatest real solution of the NBS resolvent cubic equation. The NBS algorithm provides u_1 in terms of Z_1, Z_2, Z_3, and Z_4 as follows. The formulas for q_1 and q_2 show that $q_1 + q_2 = u_1$, and the Z_n formulas show that $Z_1Z_2 = q_1$ and $Z_3Z_4 = q_2$. This demonstration therefore defines u_1 as

$$u_1 = Z_1Z_2 + Z_3Z_4 . \tag{16}$$

The Ferrari starting point includes 1) the calculation formulas for C, b_2, b_1, and b_0, 2) the Z_n-to-T_n transform, 3) the resolvent cubic equation, and 4) the T_n formulas:

$$C = A_3 / 4, \quad b_2 = A_2 - 6C^2, \quad b_1 = A_1 - 2A_2C + 8C^3, \quad b_0 = A_0 - A_1C + A_2C^2 - 3C^4, \tag{17}$$

$$Z_n = T_n - C \quad \Leftrightarrow \quad T_n = Z_n + C, \quad n = 1, 2, 3, 4,$$

$$m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0, \tag{18}$$

$$T_{1,2} = \sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 - R}, \tag{19}$$

$$T_{3,4} = -\sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 + R}, \tag{20}$$

where

$$R = \Sigma \sqrt{m^2 + b_2m + b_2^2/4 - b_0}, \tag{21}$$

$$R = b_1/\left(2\sqrt{2m}\right) \qquad (m > 0), \tag{22}$$

$$\Sigma = 1 \text{ if } b_1 > 0,\ \Sigma = -1 \text{ otherwise}, \tag{23}$$

and where $m \geq 0$ is the greatest real solution of (18).

Equations (19) and (20) provide the following preliminary relationships.

$$T_1 + T_2 = \sqrt{2m} \tag{24}$$

$$T_3 + T_4 = -\sqrt{2m} \tag{25}$$

$$T_1T_2 = m + b_2/2 + R \tag{26}$$

$$T_3T_4 = m + b_2/2 - R \tag{27}$$

Relate u_1 to m by substituting $Z_n = T_n - C$ and (24) through (27) into (16) as follows.

$$u_1 = (T_1 - C)(T_2 - C) + (T_3 - C)(T_4 - C) = T_1T_2 + T_3T_4 - (T_1+T_2+T_3+T_4)C + 2C^2$$

$$u_1 = 2m + b_2 + 2C^2 \quad \Leftrightarrow \quad 2m = u_1 - b_2 - 2C^2 \tag{28}$$

Convert the resolvent cubic equation of Ferrari to that of NBS as follows. Multiply (18) through by 8 and write as

$$(2m)^3 + 2b_2 (2m)^2 + (b_2^2 - 4b_0)(2m) - b_1^2 = 0.$$

From (28), replace 2m with $u - b_2 - 2C^2$. Substitute the expressions in (17) for C, b_2, b_1, and b_0. Then simplify to obtain the NBS resolvent cubic equation.

$$u^3 - A_2u^2 + (A_1A_3 - 4A_0)u + 4A_0A_2 - A_1^2 - A_0A_3^2 = 0 \tag{29}$$

Construct the NBS Z_n formulas from the Ferrari T_n formulas as follows. Calculate the quantities below on the left by using (24) through (27).

$$(T_1 + T_2)^2/4 - T_1T_2 = -m/2 - b_2/2 - R$$

$$(T_3 + T_4)^2/4 - T_3T_4 = -m/2 - b_2/2 + R$$

The expressions on the right are the radicands in the T_n formulas (19) and (20). These T_n formulas thus have the form:

$$T_{1,2} = (T_1 + T_2)/2 \pm\sqrt{(T_1 + T_2)^2/4 - T_1T_2}$$

$$T_{3,4} = (T_3 + T_4)/2 \pm\sqrt{(T_3 + T_4)^2/4 - T_3T_4}.$$

Obtain the corresponding formulas for the Z_n by substituting $T_n = Z_n + C$ and then simplifying.

$$Z_{1,2} = (Z_1 + Z_2)/2 \pm\sqrt{(Z_1 + Z_2)^2/4 - Z_1Z_2}$$

$$Z_{3,4} = (Z_3 + Z_4)/2 \pm\sqrt{(Z_3 + Z_4)^2/4 - Z_3Z_4}$$

These become the NBS Z_n formulas,

$$Z_{1,2} = -p_1/2 \pm \sqrt{p_1^2/4 - q_1}\,, \qquad Z_{3,4} = -p_2/2 \pm \sqrt{p_2^2/4 - q_2} \tag{30}$$

where

$$p_1 = -(Z_1+Z_2) \qquad p_2 = -(Z_3+Z_4) \tag{31}$$

and

$$q_1 = Z_1Z_2 \qquad q_2 = Z_3Z_4. \tag{32}$$

Find the NBS calculation formulas for p_1 and p_2 by substituting $T_n - C$ for the Z_n in (31), and then apply (24) or (25), (28), and (17).

$$p_1 = -(Z_1+Z_2) = 2C - (T_1 + T_2) = 2C - \sqrt{2m} = 2C - \sqrt{u_1 - b_2 - 2C^2}$$

$$p_1 = A_3/2 - \sqrt{A_3^2/4 + u_1 - A_2} \qquad p_2 = A_3/2 + \sqrt{A_3^2/4 + u_1 - A_2} \tag{33}$$

To find the NBS calculation formulas for q_1 and q_2, start by substituting $T_n - C$ for the Z_n in (32), and then apply (24) through (28).

$$q_1 = Z_1Z_2 = (T_1 - C)(T_2 - C) = T_1T_2 - (T_1 + T_2)C + C^2 = m + b_2/2 + R - C\sqrt{2m} + C^2$$

$$q_2 = Z_3Z_4 = (T_3 - C)(T_4 - C) = T_3T_4 - (T_3 + T_4)C + C^2 = m + b_2/2 - R + C\sqrt{2m} + C^2$$

or

$$q_{1,2} = m + b_2/2 + C^2 \pm \left(R - C\sqrt{2m}\right) \tag{34}$$

For the case $m > 0$, apply the formula $R = b_1/\left(2\sqrt{2m}\,\right)$ from (22). Then apply (28) for 2m.

$$q_{1,2} = m + b_2/2 + C^2 \pm \frac{b_1 - 4Cm}{2\sqrt{2m}} = u_1/2 \pm \frac{b_1 - 2Cu_1 + 2Cb_2 + 4C^3}{\sqrt{4(u_1 - b_2 - 2C^2)}} \qquad (m > 0)$$

Apply (17) for C, b_1, and b_2.

$$q_{1,2} = u_1/2 \pm \frac{A_1 - A_3 u_1/2}{\sqrt{4(u_1 - A_2 + A_3^2/4)}} \qquad (m > 0)$$

Use the NBS definition

$$\Sigma_g = \begin{cases} 1 & \text{if } A_1 - A_3 u/2 > 0 \\ -1 & \text{otherwise} \end{cases} \tag{35}$$

so that

$$A_1 - A_3 u_1/2 = \Sigma_g |A_1 - A_3 u_1/2| = \Sigma_g \sqrt{(A_1 - A_3 u_1/2)^2}$$

and $q_{1,2}$ becomes

$$q_{1,2} = u_1/2 \pm \Sigma_g \sqrt{\frac{(A_1 - A_3 u_1/2)^2}{4(u_1 - A_2 + A_3^2/4)}} = u_1/2 \pm \Sigma_g \sqrt{N/D} \qquad (m > 0) \tag{36}$$

where $N = (A_1 - A_3 u_1/2)^2$ and $D = 4(u_1 - A_2 + A_3^2/4)$.

The quotient N/D is $Q = u_1^2/4 - A_0$ because $DQ = N$, which fact we now demonstrate.

$$DQ = 4(u_1 - A_2 + A_3^2/4)(u_1^2/4 - A_0) = (u_1 - A_2 + A_3^2/4)(u_1^2 - 4A_0)$$

$$DQ = u_1^3 - A_2 u_1^2 + A_3^2 u_1^2/4 - 4A_0 u_1 + 4A_0 A_2 - A_0 A_3^2$$

Subtract zero in the form of the left side of (29) with solution u_1 replacing u. The expression for DQ becomes

$$DQ = A_1^2 - A_1 A_3 u_1 + A_3^2 u_1^2/4 = (A_1 - A_3 u_1/2)^2 = N.$$

$DQ = N$, so the quotient N/D is $Q = u_1^2/4 - A_0$. Thus, when m is greater than 0 the formulas for q_1 and q_2 in (36) become those of the NBS algorithm.

$$q_{1,2} = u_1/2 \pm \Sigma_g \sqrt{u_1^2/4 - A_0} \qquad (m > 0) \tag{37}$$

The case $m = 0$ produces this same expression for $q_{1,2}$ as now shown. Equations (34) and (21) give

$$q_{1,2} = b_2/2 + C^2 \pm \Sigma \sqrt{b_2^2/4 - b_0} \qquad (m = 0). \tag{38}$$

The resolvent cubic equation (18) implies that $b_1 = 0$, and from (23), $\Sigma = -1$. The expression for b_1 in (17) shows that

$$A_1 = 2A_2C - 8C^3 = (A_2 - A_3^2/4)A_3/2 \qquad (m = 0). \tag{39}$$

This expression and those for C, b_2, and b_0 in (17) convert $q_{1,2}$ in (38) to

$$q_{1,2} = (A_2 - A_3^2/4)/2 \pm (-1)\sqrt{(A_2 - A_3^2/4)^2/4 - A_0} \qquad (m = 0). \tag{40}$$

Equations (17), (28), (39), and (35) produce the following.

$$u_1 = b_2 + 2C^2 = A_2 - A_3^2/4, \qquad A_1 - A_3 u_1/2 = 0, \qquad \Sigma_g = -1 \qquad (m = 0)$$

The expression for $q_{1,2}$ in (40) may thus take the form of (37).

$$q_{1,2} = u_1/2 \pm \Sigma_g \sqrt{u_1^2/4 - A_0} \qquad (m = 0) \tag{41}$$

Together, (37) and (41) give the NBS q_1 and q_2 formulas for all $m \geq 0$.

$$q_1 = u_1/2 + \Sigma_g\sqrt{u_1^2/4 - A_0} \qquad q_2 = u_1/2 - \Sigma_g\sqrt{u_1^2/4 - A_0} \qquad (42)$$

This completes the construction of the NBS algorithm from the Ferrari algorithms. The NBS algorithm finds u_1 as the greatest real solution of its resolvent cubic equation (29), and then solves (35), (33), (42), and (30) in succession to find solutions Z_n of the general quartic equation. By constructing the NBS algorithm from the Ferrari algorithms, we have shown that the NBS algorithm is mathematically equivalent to the Ferrari algorithms. Because the Ferrari algorithms are mathematically equivalent to all of the algorithms described here, so is the NBS algorithm.

PART III -- Algorithm Derivations

Derivation 1: Depressed Quartic Equation

The algorithm inputs are four real coefficients A_3, A_2, A_1, and A_0, and the outputs are the four values Z_1, Z_2, Z_3 and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z.$$

The outputs are thus the four solutions of the general quartic equation

$$Z_n^4 + A_3 Z_n^3 + A_2 Z_n^2 + A_1 Z_n + A_0 = 0, \qquad n = 1, 2, 3, 4. \qquad (1\text{-}1)$$

Except for the NBS method, the algorithms solve the equivalent depressed quartic equation

$$T_n^4 + b_2T_n^2 + b_1T_n + b_0 = 0, \qquad n = 1, 2, 3, 4. \qquad (1\text{-}2)$$

The first derivation applies a shift constant C and the transform

$$Z_n = T_n - C \qquad (1\text{-}3)$$

to convert (1-1) to (1-2). Substitute (1-3) into (1-1). Expand and simplify to a quartic equation for T_n in standard form. Then equate the resulting coefficients to the corresponding coefficients in (1-2). Solve for C, b_2, b_1, and b_0. The result is

$$C = A_3/4, \quad b_2 = A_2 - 6C^2, \quad b_1 = A_1 - 2A_2C + 8C^3, \quad b_0 = A_0 - A_1C + A_2C^2 - 3C^4. \qquad (1\text{-}4)$$

Except for the NBS method, the algorithms calculate C, b_2, b_1, and b_0 in (1-4), solve (1-2) for the T_n, and apply (1-3) to compute solutions Z_n of (1-1).

Derivation 2: Ferrari Algorithms

Starting with b_2, b_1, and b_0 from (1-4) as given, Ferrari (as described by Cardano[1, pp 237-253]) finds the four solutions T_n of the depressed quartic equation (1-2). Ferrari applies an adjustable parameter m to convert (1-2) into the equality of two perfect squares

$$A^2 = B^2 \quad \Rightarrow \quad A^2 - B^2 = 0.$$

A is quadratic and B is linear in T_n. This converted quartic equation factors into two easily-solved quadratic equations:

$$A-B = 0 \quad \text{and} \quad A+B = 0.$$

The first step is to add $b_2^2/4 - b_1T_n - b_0$ to both sides of (1-2) to produce a perfect-square quartic on the left side.

$$(T_n^2 + b_2/2)^2 = b_2^2/4 - b_1T_n - b_0 \qquad (2\text{-}1)$$

The left side remains a perfect square if m is added to $T_n^2 + b_2/2$ inside the parentheses. Do this by adding $2m(T_n^2 + b_2/2) + m^2$ to both sides of (2-1). Then express the right side as a standard-form quadratic in T_n.

$$(T_n^2 + b_2/2 + m)^2 = 2mT_n^2 - b_1T_n + (m^2 + b_2m + b_2^2/4 - b_0) \qquad (2\text{-}2)$$

This equation is valid for all values of m.

The quadratic on the right side of (2-2) is a perfect square if its discriminant is zero. (The discriminant of quadratic $ax^2 + bx + c$ is $b^2 - 4ac$.) Setting the discriminate to zero produces

$$(-b_1)^2 - 4(2m)(m^2 + b_2 m + b_2^2/4 - b_0) = -8m^3 - 8b_2 m^2 - 2(b_2^2 - 4b_0)m + b_1^2 = 0.$$

Divide through by −8 to obtain Ferrari's resolvent cubic equation:

$$m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0. \qquad (2\text{-}3)$$

Any solution m of (2-3) makes the right side of (2-2) a perfect square B^2. To avoid complex-number operations, choose m as a nonnegative real solution.

$$\boxed{m \geq 0}$$

Such a solution always exists because the constant coefficient, $-b_1^2/8$, is less than or equal to zero.

With the nonnegative real solution m, equation (2-2) takes on the desired form $A^2 = B^2$:

$$(T_n^2 + b_2/2 + m)^2 = \left(\sqrt{2m}\,T_n - R\right)^2 \qquad (2\text{-}4)$$

where $$A = T_n^2 + b_2/2 + m \quad \text{and} \quad B = \sqrt{2m}\,T_n - R \qquad (2\text{-}5)$$

$$R^2 = m^2 + b_2 m + b_2^2/4 - b_0, \qquad (2\text{-}6)$$

and $$2\sqrt{2m}\,R = b_1. \qquad (2\text{-}7)$$

Equations (2-6) and (2-7) provide two different ways to solve for R. Equation (2-7) shows that R must have the same sign as b_1 if $m > 0$. If $m = 0$, then the sign of R is arbitrary in (2-4) through (2-7), but a negative R value will prove convenient later. We may therefore define the function

$$\Sigma = \begin{cases} 1 & \text{if } b_1 > 0 \\ -1 & \text{otherwise} \end{cases}, \qquad (2\text{-}8)$$

and write

$$b_1 = \Sigma|b_1| \qquad \text{and} \qquad R = \Sigma|R|. \qquad (2\text{-}9)$$

Solving (2-7) for R produces

$$R = b_1/\left(2\sqrt{2m}\right), \quad m > 0. \qquad (2\text{-}10)$$

Equations (2-6) and (2-9) imply that

$$R = \Sigma\sqrt{m^2 + b_2 m + b_2^2/4 - b_0}\,. \qquad (2\text{-}11)$$

For the case $m > 0$, (2-10) guarantees that R is a real number, so the radicand in (2-11) must be nonnegative.

$$m^2 + b_2 m + b_2^2/4 - b_0 \geq 0, \quad m > 0 \qquad (2\text{-}12)$$

With R from either (2-10) or (2-11), we proceed to factor quartic equation (2-4) into the two quadratic equations $A - B = 0$ and $A + B = 0$ using A and B from (2-5).

$$T_n^2 - \sqrt{2m}\,T_n + b_2/2 + m + R = 0 \qquad T_n^2 + \sqrt{2m}\,T_n + b_2/2 + m - R = 0 \qquad (2\text{-}13)$$

The solutions are:

$$T_{1,2} = \sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 - R} \qquad T_{3,4} = -\sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 + R}\,. \qquad (2\text{-}14)$$

The transform $Z_n = T_n - C$ then produces solutions of the general quartic equation.

$$\begin{aligned} Z_{1,2} &= \sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 - R} \\ Z_{3,4} &= -\sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 + R} \end{aligned} \qquad (2\text{-}15)$$

(Ferrari Modified Algorithm)

These are the Z_n formulas in the Ferrari modified algorithm.

The Ferrari common Z_n formulas substitute (2-10) for R into this result for the case $m > 0$.

$$\begin{aligned} Z_{1,2} &= \sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 - b_1/\left(2\sqrt{2m}\right)} \\ Z_{3,4} &= -\sqrt{m/2} - C \pm \sqrt{-m/2 - b_2/2 + b_1/\left(2\sqrt{2m}\right)} \end{aligned} \qquad (m > 0) \qquad (2\text{-}16)$$

(Ferrari Common Algorithm, $m > 0$)

Consider now the case $m = 0$. Equation (2-7) implies that b_1 must also be zero. The depressed quartic equation (1-2) becomes a quadratic equation in T_n^2 with solutions

$$T_n^2 = -b_2/2 \pm \sqrt{b_2^2/4 - b_0}\,.$$

Take the square root of both sides and apply $Z_n = T_n - C$ to obtain the Ferrari common Z_n formulas for the case $m = 0$.

$$\begin{aligned} Z_{1,2} &= -C \pm \sqrt{-b_2/2 + \sqrt{b_2^2/4 - b_0}} \\ Z_{3,4} &= -C \pm \sqrt{-b_2/2 - \sqrt{b_2^2/4 - b_0}} \end{aligned} \qquad (m = 0) \qquad (2\text{-}17)$$

(Ferrari Common Algorithm, $m = 0$)

These formulas match the results of the Ferrari modified Z_n formulas in (2-15) for $m = 0$. Equations (2-7), (2-8), and (2-11) show that $m = 0 \Rightarrow b_1 = 0 \Rightarrow \Sigma = -1 \Rightarrow R = -\sqrt{b_2^2/4 - b_0} \Rightarrow$ Z_n formulas in (2-15) for $m = 0$ match those in (2-17).

If the inner integrand $b_2^2/4 - b_0$ in (2-17) is less than zero, then the Z_n calculation requires taking square roots of complex numbers. The following paragraph demonstrates that this situation is avoided by using (2-17) only if the greatest real solution of the resolvent cubic equation is $m = 0$. If some other real solution $m > 0$ exists, then it should be used in (2-15) or (2-16) to calculate the Z_n.

Consider the case that $m = 0$ is the greatest real solution of the resolvent cubic equation (2-3). A solution $m = 0$ implies that $b_1 = 0$. The resolvent cubic equation (2-3) is therefore

$$m^3 + b_2 m^2 + (b_2^2/4 - b_0)m = m(m^2 + b_2 m + b_2^2/4 - b_0) = 0$$

with solutions 0 and $-b_2/2 \pm \sqrt{b_0}$. If $b_0 \geq 0$, then these last two solutions are real and, by assumption, less than or equal to zero.

If $b_0 \geq 0$, then $-b_2/2+\sqrt{b_0} \leq 0 \;\Rightarrow\; \sqrt{b_0} \leq b_2/2 \;\Rightarrow\; b_0 \leq b_2^2/4 \;\Rightarrow\; b_2^2/4 - b_0 \geq 0.$

Otherwise, b_0 is negative, which implies that $b_2^2/4 - b_0 \geq 0$. Therefore,

$$b_2^2/4 - b_0 \geq 0 \text{ provided no real } m > 0 \text{ exists.} \tag{2-18}$$

This result assures that (2-17) in the common algorithm operates on real numbers only when the greatest real solution of the resolvent cubic equation (2-3) is $m = 0$.

The modified algorithm applies (2-18) as well. The radicand of R in (2-11) is $b_2^2/4 - b_0$ when $m = 0$. Therefore (2-11), (2-12), and (2-18) together imply that the radicand of R is nonnegative provided that real $m > 0$ is used if it exists.

$$m^2 + b_2 m + b_2^2/4 - b_0 \geq 0 \text{ provided real } m > 0 \text{ is used if it exists.} \tag{2-19}$$

This concludes the derivation. Its results are summarized as follows.

- Given coefficients b_2, b_1, and b_0 of the depressed quartic equation (1-2), solve the resolvent cubic equation (2-3). Use a real solution $m > 0$ if it exists. Otherwise, use $m = 0$.
- For the common algorithm, calculate the Z_n using (2-16) for $m > 0$ or (2-17) for $m = 0$. Inequality (2-18) assures that (2-17) operates on real numbers only.
- For the modified algorithm, calculate Σ using (2-8), R using (2-11), and the Z_n using (2-15). By using a real $m > 0$ if it exists, the inequality (2-19) assures that R is real and that (2-15) operates on real numbers only.

Derivation 3: Descartes Algorithms

The two versions of the Descartes algorithm are similar to the corresponding versions of the Ferrari algorithm. The Ferrari formulas for Z_n become the corresponding Descartes formulas by substituting $y^2/2$ for m and positive y for $\sqrt{y^2}$. Substitute $y^2/2$ for m in the Ferrari resolvent cubic equation (2-3) and multiply through by 8 to obtain the Descartes resolvent cubic equation.

$$y^6 + 2b_2 y^4 + (b_2^2 - 4b_0)y^2 - b_1^2 = 0 \tag{3-1}$$

For the case $m = y^2/2 > 0$, the two quadratic equations in (2-13) and their solutions in (2-14) for Ferrari convert to those for Descartes by substituting (2-10) for R, $y^2/2$ for m, and positive y for $\sqrt{y^2}$.

$$T_n^2 - yT_n + (y^2 + b_2 + b_1/y)/2 = 0$$
$$T_n^2 + yT_n + (y^2 + b_2 - b_1/y)/2 = 0 \qquad (y^2 > 0) \qquad (3\text{-}2)$$

$$T_{1,2} = y/2 \pm \sqrt{-y^2/4 - b_2/2 - b_1/(2y)}$$
$$T_{3,4} = -y/2 \pm \sqrt{-y^2/4 - b_2/2 + b_1/(2y)} \qquad (y^2 > 0)$$

These T_n formulas give the solutions of the depressed quartic equation.

$$T_n^4 + b_2T_n^2 + b_1T_n + b_0 = 0, \qquad n = 1, 2, 3, 4 \qquad (3\text{-}3)$$

Rather than deriving the T_n solutions, Descartes simply asserts that the solutions T_n of the two quadratic equations (3-2) are the same as the four solutions of the depressed quartic equation (3-3) provided that y^2 is a positive real solution of the resolvent cubic equation (3-1).[5, p 184]

The following derivation verifies Descartes' assertion. Form the product of the two quadratic equations in (3-2).

$$[T_n^2 - yT_n + (y^2 + b_2 + b_1/y)/2][T_n^2 + yT_n + (y^2 + b_2 - b_1/y)/2] = 0$$

Expand and simplify to obtain

$$T_n^4 + b_2T_n^2 + b_1T_n + (y^4 + 2b_2y^2 + b_2^2 - b_1^2/y^2)/4 = 0 \qquad (3\text{-}4)$$

where y^2 is a positive real solution of (3-1). Note that equations (3-3) and (3-4) differ only in the constant coefficients. Add $4b_0y^2$ to both sides of the resolvent cubic equation (3-1), and then divide through by $4y^2$.

$$(y^4 + 2b_2y^2 + b_2^2 - b_1^2/y^2)/4 = b_0$$

This result equates the constant coefficients in (3-4) and (3-3). Thus (3-4), the product of the two quadratic equations in (3-2), is the depressed quartic equation (3-3). Therefore, the solutions of the two quadratic equations are the same as the four solutions of the depressed quartic equation provided that y^2 is a positive real solution of the resolvent cubic equation (3-1). Descartes' assertion is verified.

Derivation 4: NBS Modified Algorithm

The algorithm inputs are four real coefficients A_3, A_2, A_1, and A_0, and the outputs are the four values Z_1, Z_2, Z_3, and Z_4 such that

$$Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4) \text{ for all } Z. \qquad (4\text{-}1)$$

The outputs are thus the four solutions of the general quartic equation

$$Z_n^4 + A_3Z_n^3 + A_2Z_n^2 + A_1Z_n + A_0 = 0. \qquad (4\text{-}2)$$

The derivation comprises two sections. The first section derives the algorithm calculation equations. The second section shows that the algorithm should use the greatest real solution of the resolvent cubic equation in order to assure that the algorithm operates on real numbers only.

Calculation Equations

The NBS method expresses the left side of (4-1) as the product of two quadratics with real coefficients:

$$(Z^2+p_1Z+q_1)(Z^2+p_2Z+q_2) = Z^4 + A_3Z^3 + A_2Z^2 + A_1Z + A_0. \qquad (4\text{-}3)$$

Real values p_1, p_2, q_1, and q_2 are calculated from the coefficients A_3, A_2, A_1, and A_0, and then the solutions of quartic equation are easily computed as the roots of the two quadratics.

$$Z_{1,2} = -p_1/2 \pm \sqrt{p_1^2/4 - q_1} \qquad Z_{3,4} = -p_2/2 \pm \sqrt{p_2^2/4 - q_2} \qquad (4\text{-}4)$$

The derivation of p_1, q_1, p_2, and q_2 starts by expanding and simplifying the left side of (4-3).

$$Z^4 + (p_1+p_2)Z^3 + (p_1p_2+q_1+q_2)Z^2 + (p_1q_2+p_2q_1)Z + q_1q_2 = Z^4+A_3Z^3+A_2Z^2+A_1Z+A_0$$

Equate corresponding coefficients from the two sides to create the following system of equations.

$$p_1+p_2 = A_3 \qquad (4\text{-}5)$$

$$p_1p_2+q_1+q_2 = A_2 \qquad (4\text{-}6)$$

$$p_1q_2+p_2q_1 = A_1 \qquad (4\text{-}7)$$

$$q_1q_2 = A_0 \qquad (4\text{-}8)$$

Define u as q_1+q_2, and solve (4-6) for p_1p_2.

$$q_1+q_2 = u \qquad (4\text{-}9)$$

$$p_1p_2 = A_2 - u \qquad (4\text{-}10)$$

Apply the following fact to express p_1, p_2, q_1, and q_2 as functions of u: given the sum S and product P of two unknowns x_1 and x_2, the unknowns are found as solutions of the quadratic equation $x_m^2 - S\,x_m + P = 0$, $m = 1,2$. Equations (4-5) and (4-10) give A_3 and $A_2 - u$ as the sum and product of p_1 and p_2; (4-9) and (4-8) give u and A_0 as the sum and product of q_1 and q_2. Therefore, the quadratic equations for p_1 and p_2 and for q_1 and q_2 are

$$p_m^2 - A_3\,p_m + A_2 - u = 0 \quad \text{and} \quad q_m^2 - u\,q_m + A_0 = 0, \qquad m = 1, 2.$$

Solve the first of these by assigning the negative radical to p_1.

$$p_1 = A_3/2 - \sqrt{A_3^2/4 + u - A_2} \qquad p_2 = A_3/2 + \sqrt{A_3^2/4 + u - A_2} \qquad (4\text{-}11)$$

We cannot yet say which of q_1 or q_2 gets the positive radical and which gets the negative. For now, let Σ_g have a value of either 1 or –1, and write

$$q_1 = u/2 + \Sigma_g \sqrt{u^2/4 - A_0} \qquad q_2 = u/2 - \Sigma_g \sqrt{u^2/4 - A_0}. \qquad (4\text{-}12)$$

Equations (4-5) and (4-10) show that the radicand in (4-11) is nonnegative:

$$A_3^2/4 + u - A_2 = (p_1+p_2)^2/4 - p_1p_2 = (p_1 - p_2)^2/4 \geq 0.$$

Equations (4-9) and (4-8) show that the radicand in (4-12) is nonnegative:

$$u^2/4 - A_0 = (q_1+q_2)^2/4 - q_1q_2 = (q_1-q_2)^2/4 \geq 0.$$

To find Σ_g, substitute (4-11) and (4-12) into (4-7), and simplify to obtain

$$2\Sigma_g\sqrt{(A_3^2/4 + u - A_2)(u^2/4 - A_0)} = A_1 - A_3u/2. \qquad (4\text{-}13)$$

The radical is nonnegative, so Σ_g must correspond to the sign of the right side of the equation. We therefore express Σ_g as follows.

$$\Sigma_g = \begin{cases} 1 & \text{if } A_1 - A_3u/2 > 0 \\ -1 & \text{otherwise} \end{cases} \qquad (4\text{-}14)$$

To obtain the resolvent cubic equation in u, square both sides of (4-13) and simplify to a cubic equation in standard form:

$$u^3 - A_2u^2 + (A_1A_3 - 4A_0)u + 4A_0A_2 - A_1^2 - A_0A_3^2 = 0. \qquad (4\text{-}15)$$

At least one solution, u_1, is real and is to be used for u wherever it occurs in the calculation formulas. If multiple solutions are real, then select the greatest real solution for u_1 to assure that calculations operate on real numbers only.

This completes derivation of the algorithm calculation equations. The results are summarized as follows.

- Given the real coefficients A_3, A_2, A_1, and A_0 of the general quartic equation (4-2), solve the resolvent cubic equation (4-15). Choose the greatest real solution as u_1, and apply it as the value of u to be used hereafter.
- Calculate Σ_g using (4-14).
- Calculate p_1, p_2, q_1, and q_2 using (4-11) and (4-12).
- Calculate the quartic-equation solutions Z_1, Z_2, Z_3 and Z_4 using (4-4).

<u>Greatest Real Solution of the Resolvent Cubic Equation</u>

This section shows that the algorithm should use the greatest real solution of the resolvent cubic equation in order to assure that the algorithm operates on real numbers only. In other words, we want to assure that solution u_1 of the resolvent cubic equation and the values p_1, p_2, q_1, and q_2 are real numbers.

The resolvent cubic equation (4-15) returns three solutions for u, which is defined in (4-9) as q_1+q_2. Quantities q_1 and q_2 are the constant coefficients in the two quadratics on the left side of (4-3). Therefore, each q_m is the product of the two roots of the corresponding quadratic. If Z_1 and Z_2 are roots of the first quadratic and Z_3 and Z_4 are roots of the second quadratic, then $q_1 = Z_1Z_2$, $q_2 = Z_3Z_4$, and $u = Z_1Z_2 + Z_3Z_4$. This value of u corresponds to only one of the possible pairings of the four quartic solutions: the one in which Z_1 and Z_2 are paired together as roots of a quadratic on the left side of (4-3). Solution Z_1 can also pair with Z_3 or Z_4 as roots of a quadratic. Thus, the four quartic-equation solutions Z_n have a total of three possible pairing combinations leading to three corresponding values of u:

$$u_1 = Z_1Z_2 + Z_3Z_4 \qquad u_2 = Z_1Z_3 + Z_2Z_4 \qquad u_3 = Z_1Z_4 + Z_2Z_3. \qquad (4\text{-}16)$$

Because the different u values correspond to different pairings of the Z_n, they produce different values of p_1, p_2, q_1, and q_2.

$$u_1 \Rightarrow p_1 = -(Z_1+Z_2), \quad q_1 = Z_1Z_2, \quad p_2 = -(Z_3+Z_4), \quad q_2 = Z_3Z_4 \qquad (4\text{-}17)$$

$$u_2 \Rightarrow p_1 = -(Z_1+Z_3), \quad q_1 = Z_1Z_3, \quad p_2 = -(Z_2+Z_4), \quad q_2 = Z_2Z_4 \qquad (4\text{-}18)$$

$$u_3 \Rightarrow p_1 = -(Z_1+Z_4), \quad q_1 = Z_1Z_4, \quad p_2 = -(Z_2+Z_3), \quad q_2 = Z_2Z_3 \qquad (4\text{-}19)$$

The p_m and q_m values are real only if the two roots Z_n of a quadratic are both real or are a complex conjugate pair. At least one of the three possible u values produces such a set of real p_m and q_m values. We define u_1 as this proper choice of u, and solutions Z_1 and Z_2 as the proper roots of the first quadratic on the left side of (4-3).

The challenge is to design the algorithm to always select the proper u value for u_1. If all four of the Z_n are real, then any of the three u_k is a good choice for u_1 because all of the u_k are real and all produce real p_m and q_m values.

If the four solutions Z_n are not all real, then either 1) two of the Z_n are real and the other two are a complex conjugate pair, or 2) the four Z_n consist of two complex conjugate pairs. The u_2 and u_3 values in these cases may be real, but the corresponding p_m and q_m values may not be real. The following paragraphs consider these possibilities. Each solution Z_n is expressed as the sum of its real and imaginary components: $Z_n = X_n + iY_n$.

Suppose two of the Z_n are real and the other two are a complex conjugate pair. Let the real Z_n be roots of the first quadratic. Then $Z_1 = X_1$, $Z_2 = X_2$, $Z_3 = X_3 + iY_3$, and $Z_4 = X_3 - iY_3$ where $Y_3 > 0$. The three u values in (4-16) become

$$u_1 = X_1X_2+X_3^2+Y_3^2 \qquad u_2 = X_3(X_1+X_2) + iY_3(X_1-X_2) \qquad u_3 = X_3(X_1+X_2) - iY_3(X_1-X_2).$$

Solution u_1 is real, as are its corresponding p_m and q_m values in (4-17). If $X_1 \neq X_2$, then u_2 and u_3 are a complex conjugate pair with nonzero imaginary components. Such complex u values are avoided, and the real u value is selected as u_1. If $X_1 = X_2$, then the three u values are all real, but u_1 is greater than u_2 and u_3:

$$u_1 = X_1^2+X_3^2+Y_3^2 \qquad u_2 = u_3 = 2X_1X_3,$$

$$u_1 = X_1^2+X_3^2+Y_3^2 > X_1^2+X_3^2 = (X_1-X_3)^2 + 2X_1X_3 \geq 2X_1X_3 = u_2 = u_3 \qquad \therefore u_1 > u_2 = u_3.$$

Although u_2 and u_3 are real, they should be avoided because their corresponding p_m and q_m values in (4-18) and (4-19) are complex. For example, p_1 in (4-18) becomes

$$p_1 = -(Z_1+Z_3) = -(X_1 + X_3 + iY_3) = -(X_1 + X_3) - iY_3 \quad \text{where} \quad Y_3 > 0.$$

Using u_1 as the greatest real solution of the resolvent cubic equation avoids such complex p_m and q_m values.

Now suppose that the four Z_n consist of two complex conjugate pairs. Then $Z_1 = X_1+iY_1$, $Z_2 = X_1-iY_1$, $Z_3 = X_3 + iY_3$, and $Z_4 = X_3 - iY_3$ where $Y_1 > 0$ and $Y_3 > 0$. The three u values in (4-16) become

$$u_1 = X_1^2+Y_1^2+X_3^2+Y_3^2 \qquad u_2 = 2(X_1X_3-Y_1Y_3) \qquad u_3 = 2(X_1X_3+Y_1Y_3).$$

All three u values are real, but u_1 is at least as great as u_2 and u_3.

$$u_1 = X_1^2+Y_1^2+X_3^2+Y_3^2 = (X_1-X_3)^2+2X_1X_3+(Y_1-Y_3)^2+2Y_1Y_3$$

$$\geq 2X_1X_3 + 2Y_1Y_3 = u_3 > 2X_1X_3 - 2Y_1Y_3 = u_2.$$

$$\therefore u_1 \geq u_3 > u_2.$$

The p_m and q_m values corresponding to u_1 in (4-17) are real, but those corresponding to u_2 in (4-18) are complex. The p_m and q_m values corresponding to u_3 in (4-19) are also complex unless $Z_1=Z_3$. In any case, complex p_m and q_m values are avoided by selecting u_1 as the greatest of the three real solutions of the resolvent cubic equation.

We see that selecting u_1 as the greatest real solution of the resolvent cubic equation for all cases assures that the algorithm operates only on real numbers: u_1, p_1, p_2, q_1, and q_2.

Derivation 5: Euler Algorithms

The Euler algorithms solve the depressed quartic equation

$$T_n^4 + b_2T_n^2 + b_1T_n + b_0 = 0, \qquad n = 1, 2, 3, 4. \qquad (5\text{-}1)$$

Euler's derivation[7, pp 256-257] assumes that a solution of some quartic equation has the form

$$T_n = \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} \qquad (5\text{-}2)$$

where r_1, r_2, and r_3 are the three solutions of a cubic equation with real coefficients:

$$r_k^3 + a_2r_k^2 + a_1r_k + a_0 = 0. \qquad (5\text{-}3)$$

The derivation finds that the quartic equation with solution (5-2) is a depressed quartic equation (5-1) whose coefficients b_n are functions of the a_m in (5-3). The derivation inverts these functions to calculate the a_m from b_n. The cubic equation (5-3) with coefficients expressed in terms of the b_n becomes the resolvent cubic equation. Its solutions r_1, r_2, and r_3 enable calculation of T_n via (5-2).

Euler does not employ the principal-square-root convention for radicals. Instead, he initially allows each of the $\sqrt{r_k}$ in (5-2) to be either square root of r_k. The two square-root values for each of the three r_k provide eight combinations of square-root terms in (5-2) for eight possible values for T_n. The derivation shows that only four of these eight are valid for any given b_1 value in (5-1).

The derivation begins with the requirement for the r_k in (5-3):

$$(r - r_1)(r - r_2)(r - r_3) = r^3 + a_2r^2 + a_1r + a_0 \text{ for all r.}$$

Expand and simplify the left side. Then equate corresponding coefficients on the two sides to obtain:

$$-(r_1+r_2+r_3) = a_2 \qquad (5\text{-}4)$$

$$r_1r_2+r_1r_3+r_2r_3 = a_1 \qquad (5\text{-}5)$$

$$-r_1r_2r_3 = a_0. \qquad (5\text{-}6)$$

We proceed to find the quartic equation whose solutions are given by (5-2). First square (5-2). Then apply (5-4) and rearrange.

$$T_n^2 = r_1 + r_2 + r_3 + 2\sqrt{r_1r_2} + 2\sqrt{r_1r_3} + 2\sqrt{r_2r_3}$$

$$T_n^2 + a_2 = 2\left(\sqrt{r_1r_2} + \sqrt{r_1r_3} + \sqrt{r_2r_3}\right)$$

Square both sides to obtain

$$T_n^4 + 2a_2T_n^2 + a_2^2 = 4(r_1r_2 + r_1r_3 + r_2r_3) + 8\sqrt{r_1}\sqrt{r_2}\sqrt{r_3}\left(\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}\right).$$

Apply (5-5) and (5-2). Then rearrange to standard form.

$$T_n^4 + 2a_2T_n^2 + a_2^2 = 4a_1 + 8\sqrt{r_1}\sqrt{r_2}\sqrt{r_3}\,T_n$$

$$T_n^4 + 2a_2T_n^2 - 8\sqrt{r_1}\sqrt{r_2}\sqrt{r_3}\,T_n + a_2^2 - 4a_1 = 0 \qquad (5\text{-}7)$$

Equation (5-6) shows that

$$\sqrt{r_1}\sqrt{r_2}\sqrt{r_3} = \sqrt{r_1r_2r_3} = \sqrt{-a_0}. \qquad (5\text{-}8)$$

Equation (5-7) becomes

$$T_n^4 + 2a_2T_n^2 - 8\sqrt{-a_0}\,T_n + a_2^2 - 4a_1 = 0, \qquad (5\text{-}9)$$

which has the form of (5-1).

With this information, we can now find the resolvent cubic equation for the depressed quartic equation (5-1). Equate coefficients b_n in (5-1) to the corresponding coefficients in (5-9).

$$b_2 = 2a_2 \qquad b_1 = -8\sqrt{-a_0} \qquad b_0 = a_2^2 - 4a_1. \qquad (5\text{-}10)$$

Solve this system of equations for a_2, a_1, and a_0, and apply the results to (5-3).

$$a_2 = b_2/2 \qquad a_1 = (b_2^2 - 4b_0)/16 \qquad a_0 = -b_1^2/64 \qquad (5\text{-}11)$$

$$r_k^3 + (b_2/2)r_k^2 + [(b_2^2 - 4b_0)/16]r_k - b_1^2/64 = 0 \qquad (5\text{-}12)$$

This is Euler's resolvent cubic equation.

The constant coefficient, $a_0 = -b_1^2/64$, provides information about the equation's three solutions, r_1, r_2, and r_3. Because $-b_1^2/64$ is less than or equal to zero, the equation has at least one nonnegative real solution, say r_1: $r_1 \geq 0$. Equations (5-6) and (5-11) for a_0 combine to show that

$$r_1r_2r_3 = b_1^2/64 \geq 0. \qquad (5\text{-}13)$$

The product of all three solutions is a nonnegative real number. As a result, the product r_2r_3 is a nonnegative real number:

$$r_1 \geq 0 \quad \text{and} \quad r_1r_2r_3 = b_1^2/64 \geq 0 \quad \Rightarrow \quad r_2r_3 \geq 0.$$

Solutions r_2 and r_3 are real or they form a complex conjugate pair. If they are real, then they cannot have opposite signs.

Through its constant coefficient, $-b_1^2/64$, equation (5-12) depends on the modulus of b_1, but not on its sign. Thus (5-12) is the resolvent cubic equation for two quartic equations:

$$T_n^4 + b_2T_n^2 \pm b_1T_n + b_0 = 0.$$

Each of these two quartic equations has four solutions for a total of eight solutions. These are the eight possible values of T_n given by (5-2):

$$T_n = \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3}\,. \qquad (5\text{-}2)$$

To determine which four of the T_n values from (5-2) are solutions of

$$T_n^4 + b_2T_n^2 + b_1T_n + b_0 = 0, \qquad (5\text{-}1)$$

combine (5-10) for b_1 with (5-8) to obtain

$$\sqrt{r_1}\sqrt{r_2}\sqrt{r_3} = -b_1/8. \qquad (5\text{-}14)$$

Equation (5-2) produces a desired solution T_n only if the three radical terms on the right satisfy (5-14). The user is allowed to select either of two square roots for any two of the $\sqrt{r_k}$. The third $\sqrt{r_k}$ is selected to satisfy (5-14). The user only needs to check that the two sides of (5-14) have the same sign because (5-13) guarantees that they have the same modulus.

$$r_1r_2r_3 = b_1^2/64 \quad \Rightarrow \quad \left|\sqrt{r_1}\sqrt{r_2}\sqrt{r_3}\right| = |b_1/8|$$

With a set of square roots $\sqrt{r_k}$ that satisfies (5-14), the four solutions of (5-1) become:

$$T_1 = \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} \qquad (5\text{-}15)$$

$$T_2 = \sqrt{r_1} - \sqrt{r_2} - \sqrt{r_3} \qquad (5\text{-}16)$$

$$T_3 = -\sqrt{r_1} + \sqrt{r_2} - \sqrt{r_3} \qquad (5\text{-}17)$$

$$T_4 = -\sqrt{r_1} - \sqrt{r_2} + \sqrt{r_3}\,. \qquad (5\text{-}18)$$

Each of these T_n expressions is valid because its terms are all square roots of the r_k, and the product of its terms equals $-b_1/8$ to satisfy (5-14).

In summary, Euler's original algorithm

- starts with the coefficients b_2, b_1, and b_0 of the depressed quartic equation (5-1),
- solves the resolvent cubic equation (5-12) for r_1, r_2, and r_3,

- selects signs of $\sqrt{r_1}$, $\sqrt{r_2}$, and $\sqrt{r_3}$ to satisfy (5-14),
- and calculates the four solutions T_n of (5-1) by using (5-15) through (5-18).

The conversion of Euler's original algorithm to the modified algorithm was described previously in Part II. Titles of the relevant Part II sections are underlined in the following summary. The original T_n formulas are recast to use the Principal-Square-Root Convention for Radicals, equations (6) and (7). Solution r_1 of the resolvent cubic equation is defined as the Greatest Real Solution of the Resolvent Cubic Equation. The section T_n Formulas for Euler and Van der Waerden Algorithms shows that r_1 is nonnegative, (13), and derives the simplified form of the modified T_n formulas, (15).

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{r_2 + r_3 - 2\Sigma\sqrt{r_2 r_3}} \qquad T_{3,4} = -\sqrt{r_1} \pm \sqrt{r_2 + r_3 + 2\Sigma\sqrt{r_2 r_3}}.$$

The inner integrand r_2r_3 is a nonnegative real number by (13). Solutions $r_2 = x_2 + iy_2$ and $r_3 = x_3 + iy_3$ of the resolvent cubic equation are real ($y_2 = y_3 = 0$), or they form a complex conjugate pair ($x_2 = x_3$, $y_2 = -y_3 > 0$). In either case, the sum $r_2 + r_3$ equals $x_2 + x_3$, and the product r_2r_3 equals $x_2x_3 + y_2^2$. The modified algorithm T_n formulas become

$$T_{1,2} = \sqrt{r_1} \pm \sqrt{x_2 + x_3 - 2\Sigma\sqrt{x_2x_3 + y_2^2}} \quad T_{3,4} = -\sqrt{r_1} \pm \sqrt{x_2 + x_3 + 2\Sigma\sqrt{x_2x_3 + y_2^2}}.$$

All constituents of these T_n formulas are real numbers, and the inner integrand $r_2r_3 = x_2x_3 + y_2^2$ is a nonnegative real number. Thus, the T_n formulas require operations on real numbers only.

Derivation 6: Van der Waerden Algorithms

This section derives the Van der Waerden original and modified algorithms for solving the depressed quartic equation

$$T_n^4 + b_2T_n^2 + b_1T_n + b_0 = 0, \qquad n = 1, 2, 3, 4. \tag{6-1}$$

In the Van der Waerden original algorithm and its derivation here, the radical indicates that either of the two possible square roots applies.

The Resolvent Cubic Equation

Express the quartic in (6-1) as the product of two quadratics with real coefficients.

$$(T^2+p_1T+q_1)(T^2+p_2T+q_2) = T^4 + b_2T^2 + b_1T + b_0 \tag{6-2}$$

Expand and simplify the left side.

$$T^4 + (p_1+p_2)T^3 + (p_1p_2+q_1+q_2)T^2 + (p_1q_2+p_2q_1)T + q_1q_2 = T^4 + b_2T^2 + b_1T + b_0$$

Equate corresponding coefficients from the two sides to create the following system of equations.

$$p_1+p_2 = 0 \tag{6-3}$$

$$p_1p_2+q_1+q_2 = b_2 \tag{6-4}$$

$$p_1q_2+p_2q_1 = b_1 \tag{6-5}$$

$$q_1 q_2 = b_0 \tag{6-6}$$

Define θ_k as $$\theta_k = p_1 p_2. \tag{6-7}$$

Then (6-4) becomes $$q_1 + q_2 = b_2 - \theta_k. \tag{6-8}$$

Equations (6-3) and (6-7) give 0 and θ_k as the sum and product of p_1 and p_2; (6-8) and (6-6) give $b_2 - \theta_k$ and b_0 as the sum and product of q_1 and q_2. Values of p_1, p_2, q_1, and q_2 are therefore solutions of the quadratic equations

$$p_m^2 + \theta_k = 0 \qquad \text{and} \qquad q_m^2 - (b_2 - \theta)q_m + b_0 = 0, \qquad m = 1, 2.$$

The solutions are

$$p_1 = -\sqrt{-\theta_k} \qquad\qquad q_1 = \tfrac{1}{2}\left[b_2 - \theta_k + \Sigma'\sqrt{(b_2 - \theta_k)^2 - 4b_0}\right]$$

$$p_2 = \sqrt{-\theta_k} \qquad\qquad q_2 = \tfrac{1}{2}\left[b_2 - \theta_k - \Sigma'\sqrt{(b_2 - \theta_k)^2 - 4b_0}\right]$$

where Σ' has a value of either 1 or –1.

Substitute the expressions for p_1, p_2, q_1, and q_2 into (6-5) and simplify.

$$\Sigma'\sqrt{-\theta_k[(b_2 - \theta_k)^2 - 4b_0]} = b_1$$

Square both sides and rearrange to form the Van der Waerden resolvent cubic equation.

$$\theta_k^3 - 2b_2\theta_k^2 + (b_2^2 - 4b_0)\theta_k + b_1^2 = 0 \tag{6-9}$$

The Three Solutions of the Resolvent Cubic Equation

The three solutions θ_1, θ_2, and θ_3 of (6-9) satisfy the requirement

$$(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3) = \theta^3 - 2b_2\theta^2 + (b_2^2 - 4b_0)\theta + b_1^2 \quad \text{for all } \theta.$$

Expand and simplify the left side, and then equate corresponding coefficients from the two sides.

$$\theta_1 + \theta_2 + \theta_3 = 2b_2 \tag{6-10}$$

$$\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = b_2^2 - 4b_0 \tag{6-11}$$

$$-\theta_1\theta_2\theta_3 = b_1^2 \tag{6-12}$$

Van der Waerden uses all three solutions θ_1, θ_2, and θ_3 of (6-9). Each solution corresponds to its particular grouping of the four T_n into two pair: each pair of T_n are the roots of a quadratic on the left side (6-2).

Solution θ_1 corresponds to T_1 paired with T_2 as roots of $T^2+p_1T+q_1$, so T_3 and T_4 are roots of $T^2+p_2T+q_2$. Then

$$(T - T_1)(T - T_2) = T^2 + p_1T + q_1 \quad \text{and} \quad (T - T_3)(T - T_4) = T^2 + p_2T + q_2 \qquad \text{imply that}$$

$$p_1 = -(T_1 + T_2) \qquad p_2 = -(T_3 + T_4).$$

These expressions for p_1 and p_2 combined with (6-3) and (6-7) show that

$$T_1 + T_2 + T_3 + T_4 = 0 \qquad (6\text{-}13)$$

and
$$\theta_1 = (T_1 + T_2)(T_3 + T_4) = -(T_1 + T_2)^2 = -(T_3 + T_4)^2. \qquad (6\text{-}14)$$

Solutions θ_2 and θ_3 correspond to the two alternate pairings: T_1 paired with T_3 and T_1 paired with T_4:

$$\theta_2 = (T_1 + T_3)(T_2 + T_4) = -(T_1 + T_3)^2 = -(T_2 + T_4)^2 \qquad (6\text{-}15)$$

$$\theta_3 = (T_1 + T_4)(T_2 + T_3) = -(T_1 + T_4)^2 = -(T_2 + T_3)^2. \qquad (6\text{-}16)$$

Solutions of the Depressed Quartic Equation in the Original Algorithm

The solutions T_n of the depressed quartic equation derive from (6-13) to (6-16). Equations (6-14) to (6-16) provide the following corresponding expressions.

$$T_1 + T_2 = -(T_3 + T_4) = \sqrt{-\theta_1}$$

$$T_1 + T_3 = -(T_2 + T_4) = \sqrt{-\theta_2}$$

$$T_1 + T_4 = -(T_2 + T_3) = \sqrt{-\theta_3}.$$

By invoking $T_1+T_2+T_3+T_4 = 0$ from (6-13), these last three equations convert to the four T_n formulas as follows.

$$T_1 = \tfrac{1}{2}(T_1+T_2 + T_1+T_3 + T_1+T_4) \Rightarrow \boxed{T_1 = \tfrac{1}{2}\left(\sqrt{-\theta_1} + \sqrt{-\theta_2} + \sqrt{-\theta_3}\right)} \qquad (6\text{-}17)$$

$$T_2 = \tfrac{1}{2}(T_1+T_2 + T_2+T_4 + T_2+T_3) \Rightarrow \boxed{T_2 = \tfrac{1}{2}\left(\sqrt{-\theta_1} - \sqrt{-\theta_2} - \sqrt{-\theta_3}\right)} \qquad (6\text{-}18)$$

$$T_3 = \tfrac{1}{2}(T_3+T_4 + T_1+T_3 + T_2+T_3) \Rightarrow \boxed{T_3 = \tfrac{1}{2}\left(-\sqrt{-\theta_1} + \sqrt{-\theta_2} - \sqrt{-\theta_3}\right)} \qquad (6\text{-}19)$$

$$T_4 = \tfrac{1}{2}(T_3+T_4 + T_2+T_4 + T_1+T_4) \Rightarrow \boxed{T_4 = \tfrac{1}{2}\left(-\sqrt{-\theta_1} - \sqrt{-\theta_2} + \sqrt{-\theta_3}\right)} \qquad (6\text{-}20)$$

To qualify as solutions of the depressed quartic equation (6-1), these expressions for the four T_n must satisfy the requirement

$$(T-T_1)(T-T_2)(T-T_3)(T-T_4) = T^4 + b_2T^2 + b_1T + b_0 \quad \text{for all } T.$$

That is, the T_n must satisfy the system:

$$T_1 + T_2 + T_3 + T_4 = 0 \qquad (6\text{-}21)$$

$$T_1T_2 + T_1T_3 + T_1T_4 + T_2T_3 + T_2T_4 + T_3T_4 = b_2 \qquad (6\text{-}22)$$

$$-T_1T_2T_3 - T_1T_2T_4 - T_1T_3T_4 - T_2T_3T_4 = b_1 \qquad (6\text{-}23)$$

$$T_1T_2T_3T_4 = b_0. \qquad (6\text{-}24)$$

The T_n of (6-17) to (6-20) do satisfy (6-21). They also satisfy (6-22) and (6-24) as verified with (6-10) and (6-11). However, (6-23) holds only if

$$\boxed{\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3} = -b_1. \qquad (6\text{-}25)}$$

Equation (6-25) becomes a restriction on the $\sqrt{-\theta_k}$. The user selects either square root for any two of the $\sqrt{-\theta_k}$. The third $\sqrt{-\theta_k}$ is selected to satisfy (6-25). The user only needs to check that the two sides of this equation have the same sign. Equation (6-12) guarantees that the two sides have equal magnitudes:

$$-\theta_1\theta_2\theta_3 = b_1^2 \quad \Rightarrow \quad \left|\sqrt{-\theta_1\theta_2\theta_3}\right| = \left|\sqrt{-\theta_1}\,\sqrt{-\theta_2}\,\sqrt{-\theta_3}\right| = |-b_1|. \tag{6-26}$$

This completes the derivation of the Van der Waerden original algorithm, which is summarized as follows.

- Given the coefficients b_2, b_1, and b_0 of (6-1), the algorithm solves the resolvent cubic equation (6-9) for its three solutions θ_1, θ_2, and θ_3.
- Signs of the three $\sqrt{-\theta_k}$ are selected to satisfy (6-25).
- Equations (6-17) to (6-20) give the solutions T_1, T_2, T_3 and T_4 of the depressed quartic equation (6-1).

Van der Waerden Modified Algorithm

Conversion of the Van der Waerden original algorithm to the modified algorithm is similar to the corresponding conversion involving the Euler algorithms.

First, the original T_n formulas are recast to use the principal-square-root convention for radicals. This is accomplished by replacing $\sqrt{-\theta_3}$ with $-\Sigma s\sqrt{-\theta_3}$ where

$$\Sigma = \begin{cases} 1 & \text{if } b_1 > 0 \\ -1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad s = \begin{cases} 1 & \text{if } \sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3} \geq 0 \\ -1 & \text{otherwise.} \end{cases} \tag{6-27}$$

The definitions of these special functions and (6-26) imply that

$$b_1 = \Sigma|b_1| \qquad \text{and} \qquad \sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3} = s\left|\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3}\right| = s|b_1|.$$

The Van der Waerden original T_n formulas change

from $T_{1,2} = \frac{1}{2}\left[\sqrt{-\theta_1} \pm \left(\sqrt{-\theta_2} + \sqrt{-\theta_3}\right)\right]$ and $T_{3,4} = \frac{1}{2}\left[-\sqrt{-\theta_1} \pm \left(\sqrt{-\theta_2} - \sqrt{-\theta_3}\right)\right]$ to

$$T_{1,2} = \frac{1}{2}\left[\sqrt{-\theta_1} \pm \left(\sqrt{-\theta_2} - \Sigma s\sqrt{-\theta_3}\right)\right] \qquad T_{3,4} = \frac{1}{2}\left[-\sqrt{-\theta_1} \pm \left(\sqrt{-\theta_2} + \Sigma s\sqrt{-\theta_3}\right)\right]. \tag{6-28}$$

In this revised formulation, the product of terms inside the brackets for all T_n is

$$-\Sigma s\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3} = -\Sigma s^2|b_1| = -\Sigma|b_1| = -b_1 \qquad \text{as required by (6-25).}$$

The function s in (6-27) accommodates the condition $\sqrt{-\theta_1}\sqrt{-\theta_2}\sqrt{-\theta_3} < 0$, which occurs when one of the $-\theta_k$, say $-\theta_1$, is positive real and the other two $-\theta_k$ are negative real:
$\sqrt{-\theta_2} = i\sqrt{|\theta_2|}$, $\sqrt{-\theta_3} = i\sqrt{|\theta_3|} \quad \Rightarrow \quad \sqrt{-\theta_2}\sqrt{-\theta_3} = -\sqrt{|\theta_2|}\sqrt{|\theta_3|} = -\sqrt{\theta_2\theta_3} < 0.$

The revised Van der Waerden algorithm specifies that θ_1 is a nonpositive real solution, $\theta_1 \le 0$, of the resolvent cubic equation (6-9). Such a solution must exist because the constant term on the left side is $b_1^2 \ge 0$. As a result, both $-\theta_1$ and its principal square root are nonnegative real numbers.

$$-\theta_1 \ge 0 \quad \Rightarrow \quad \sqrt{-\theta_1} \ge 0 \tag{6-29}$$

Equation (6-12) shows that $-\theta_1\theta_2\theta_3 = b_1^2 \ge 0$. Therefore,

$$-\theta_1 \ge 0 \quad \text{and} \quad -\theta_1\theta_2\theta_3 = b_1^2 \ge 0 \quad \Rightarrow \quad \theta_2\theta_3 \ge 0. \tag{6-30}$$

The product $\theta_2\theta_3$ is a nonnegative real number. Thus, $\theta_2 = \theta_{x2} + i\theta_{y2}$ and $\theta_3 = \theta_{x3} + i\theta_{y3}$ are real ($\theta_{y2} = \theta_{y3} = 0$), or they form a complex conjugate pair ($\theta_{x2} = \theta_{x3}$, $\theta_{y2} = -\theta_{y3} > 0$). If real, then they cannot have opposite signs. This restriction on θ_2 and θ_3 implies that each parenthetical expression in (6-28) is either real or pure imaginary.

Conversion of the T_n formulas in (6-28) to those in the Van der Waerden modified algorithm starts by replacing each parenthetical expression with the radical of its square.

$$\begin{aligned} T_{1,2} &= \tfrac{1}{2}\left[\ \ \sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 - 2s\Sigma\sqrt{-\theta_2}\sqrt{-\theta_3}}\ \right] \\ T_{3,4} &= \tfrac{1}{2}\left[-\sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 + 2s\Sigma\sqrt{-\theta_2}\sqrt{-\theta_3}}\ \right] \end{aligned} \tag{6-31}$$

T_1 and T_2 in (6-31) each have the same value as in (6-28) unless $\sqrt{-\theta_2} - \Sigma s\sqrt{-\theta_3}$ happens to be either negative real or negative imaginary. In that case, T_1 in (6-28) becomes T_2 in (6-31) and T_2 in (6-28) becomes T_1 in (6-31). T_3 and T_4 are correspondingly affected by the value of $\sqrt{-\theta_2} + \Sigma s\sqrt{-\theta_3}$.

As an option to prevent the T_n from flipping values between (6-28) and (6-31), use the following convention: select $\theta_2 = \theta_{x2} + i\theta_{y2}$ and $\theta_3 = \theta_{x3} + i\theta_{y3}$ so that $|\theta_{x2}| \ge |\theta_{x3}|$ and $\theta_{y2} = -\theta_{y3} \ge 0$. The convention assures that the parenthetical expressions in (6-28) are either nonnegative real or nonnegative imaginary. The convention does not affect (6-31), which is symmetrical with respect to θ_2 and θ_3.

Equation (6-29) implies that the formula for s in (6-27) simplifies to

$$s = \begin{cases} 1 & \text{if } \sqrt{-\theta_2}\sqrt{-\theta_3} \ge 0 \\ -1 & \text{otherwise} \end{cases} \quad \Rightarrow \quad \sqrt{-\theta_2}\sqrt{-\theta_3} = s\left|\sqrt{-\theta_2}\sqrt{-\theta_3}\right| = s\left|\sqrt{\theta_2\theta_3}\right|.$$

This result and (6-30) imply that

$$s\sqrt{-\theta_2}\sqrt{-\theta_3} = s^2\left|\sqrt{\theta_2\theta_3}\right| = \sqrt{\theta_2\theta_3}\,.$$

The T_n formulas in (6-31) become

$$T_{1,2} = \tfrac{1}{2}\left[\sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 - 2\Sigma\sqrt{\theta_2\theta_3}}\ \right] \qquad T_{3,4} = \tfrac{1}{2}\left[-\sqrt{-\theta_1} \pm \sqrt{-\theta_2 - \theta_3 + 2\Sigma\sqrt{\theta_2\theta_3}}\ \right].$$

Whether the values $\theta_2 = \theta_{x2} + i\theta_{y2}$ and $\theta_3 = \theta_{x3} + i\theta_{y3}$ are real ($\theta_{y2} = \theta_{y3} = 0$) or form a complex conjugate pair ($\theta_{x2} = \theta_{x3}$, $\theta_{y2} = -\theta_{y3} > 0$), we have $-\theta_2 - \theta_3 = -\theta_{x2} - \theta_{x3}$ and $\theta_2\theta_3 = \theta_{x2}\theta_{x3} + \theta_{y2}^2$. The final T_n formulas for the Van der Waerden modified algorithm become

$$T_{1,2} = \frac{1}{2}\left[\sqrt{-\theta_1} \pm \sqrt{-\theta_{x2} - \theta_{x3} - 2\Sigma\sqrt{\theta_{x2}\theta_{x3} + \theta_{y2}^2}} \right]$$

$$T_{3,4} = \frac{1}{2}\left[-\sqrt{-\theta_1} \pm \sqrt{-\theta_{x2} - \theta_{x3} + 2\Sigma\sqrt{\theta_{x2}\theta_{x3} + \theta_{y2}^2}} \right].$$

All constituents of these Van der Waerden modified T_n formulas are real numbers, and the inner integrand $\theta_2\theta_3 = \theta_{x2}\theta_{x3} + \theta_{y2}^2$ is a nonnegative real number. The calculation therefore requires operations on real numbers only.

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Appendix A -- Computational Instability of the Ferrari Common Algorithm

This appendix demonstrates that the Ferrari common algorithm becomes computationally unstable as solution m of the resolvent cubic equation approaches zero. The table below gives the algorithm to find the four solutions T_n (n = 1,2,3,4) of the depressed quartic equation

$$T_n^4 + b_2 T_n^2 + b_1 T_n + b_0 = 0 \qquad \text{(A-1)}$$

where the three real coefficients b_2, b_1, and b_0 are given.

<table>
<tr><td colspan="2">Solve this resolvent cubic equation for real m:
$m^3 + b_2 m^2 + (b_2^2/4 - b_0)m - b_1^2/8 = 0.$ (A-2)
Use a real solution m > 0 if it exists. Otherwise, m = 0.</td></tr>
<tr><td>If m > 0, then
$T_{1,2} = \sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 - b_1/(2\sqrt{2m})}$
$T_{3,4} = -\sqrt{m/2} \pm \sqrt{-m/2 - b_2/2 + b_1/(2\sqrt{2m})}$</td><td>If m = 0, then
$T_{1,2} = \pm\sqrt{-b_2/2 + \sqrt{b_2^2/4 - b_0}}$
$T_{3,4} = \pm\sqrt{-b_2/2 - \sqrt{b_2^2/4 - b_0}}$</td></tr>
</table>

If a nonzero real solution m of the resolvent cubic equation becomes sufficiently small, then the m^3 and m^2 terms vanish, and that equation becomes

$$(b_2^2/4 - b_0)m - b_1^2/8 = 0 \quad \Rightarrow \quad b_1^2/(8m) = b_2^2/4 - b_0 \quad \Rightarrow$$

$$b_1 = \pm\left(2\sqrt{2m}\right)\sqrt{b_2^2/4 - b_0} \qquad \text{for sufficiently small m.}$$

Thus, b_1 approaches zero as m approaches zero.

The fraction $b_1/\left(2\sqrt{2m}\right)$ causes computational instability for small m because the m value, calculated as the solution of a cubic equation, typically contains a small round-off error e not found in b_1. Square brackets distinguish such a calculated value [m] from the true value m.

$$[m] = m + e \qquad \text{(A-3)}$$

The calculated fraction $\left[b_1/\left(2\sqrt{2m}\right)\right]$ becomes

$$\left[\frac{b_1}{2\sqrt{2m}}\right] = \frac{b_1}{2\sqrt{2[m]}} = \frac{b_1}{2\sqrt{2(m+e)}} = \frac{b_1}{2\sqrt{2m}}\sqrt{\frac{m}{m+e}} = \frac{b_1}{2\sqrt{2m}}(1 + e/m)^{-1/2}.$$

When m approaches the magnitude of e, then the calculated fraction $\left[b_1/\left(2\sqrt{2m}\right)\right]$ is dominated by error. Suppose e is negative. As m approaches |e|, the factor $(1 + e/m)^{-1/2}$ becomes unbounded. The calculated fraction $\left[b_1/\left(2\sqrt{2m}\right)\right]$ and the calculated solutions $[T_n]$ are then also unbounded.

The Ferrari common algorithm becomes unstable when m and b_1 approach the order of round-off error e, but the instability can be even worse as shown in the following example. Let $\mu > 0$ be an adjustable real parameter, and let the true solutions T_n of a depressed quartic equation be

$$T_1 = T_2 = \sqrt{\mu/2} \quad \text{and} \quad T_{3,4} = -\sqrt{\mu/2} \pm 2i. \qquad \text{(A-4)}$$

The depressed quartic equation (A-1) is $(T_n - T_1)(T_n - T_2)(T_n - T_3)(T_n - T_4) =$

$$\left(T_n - \sqrt{\mu/2}\right)^2 [T_n - (-\sqrt{\mu/2} + 2i\,]\,[T_n - (-\sqrt{\mu/2} - 2i\,)] =$$

$$\boxed{T_n^4 + (4-\mu)T_n^2 - 8\sqrt{\mu/2}\,T_n + 2\mu + \mu^2/4 = 0.}$$

The coefficients b_2, b_1, and b_0 are:

$$b_2 = 4 - \mu \qquad b_1 = -8\sqrt{\mu/2} \qquad b_0 = 2\mu + \mu^2/4. \qquad \text{(A-5)}$$

The resolvent cubic equation (A-2) is

$$m^3 + (4-\mu)m^2 + 4(1-\mu)m - 4\mu = 0. \qquad \text{(A-6)}$$

The left side factors to $(m-\mu)(m+2)^2$, so the three solutions are μ, -2, and -2. Solution $m = \mu$, the only nonnegative real solution, applies. If (A-5) and $m = \mu$ are inserted into the Ferrari common T_n formulas, those formulas produce the four T_n solutions in (A-4).

If the coefficients in (A-6) are given as numerical values, then the calculated solution $[m] = m + e = \mu + e$ contains a round-off error e, which in turn produces error in the calculated values of the T_n. The calculated solution for T_1 using the Ferrari common algorithm is

$$[T_1] = \sqrt{[m]/2} + \sqrt{-[m]/2 - b_2/2 - b_1/\left(2\sqrt{2[m]}\,\right)}. \qquad \text{(A-7)}$$

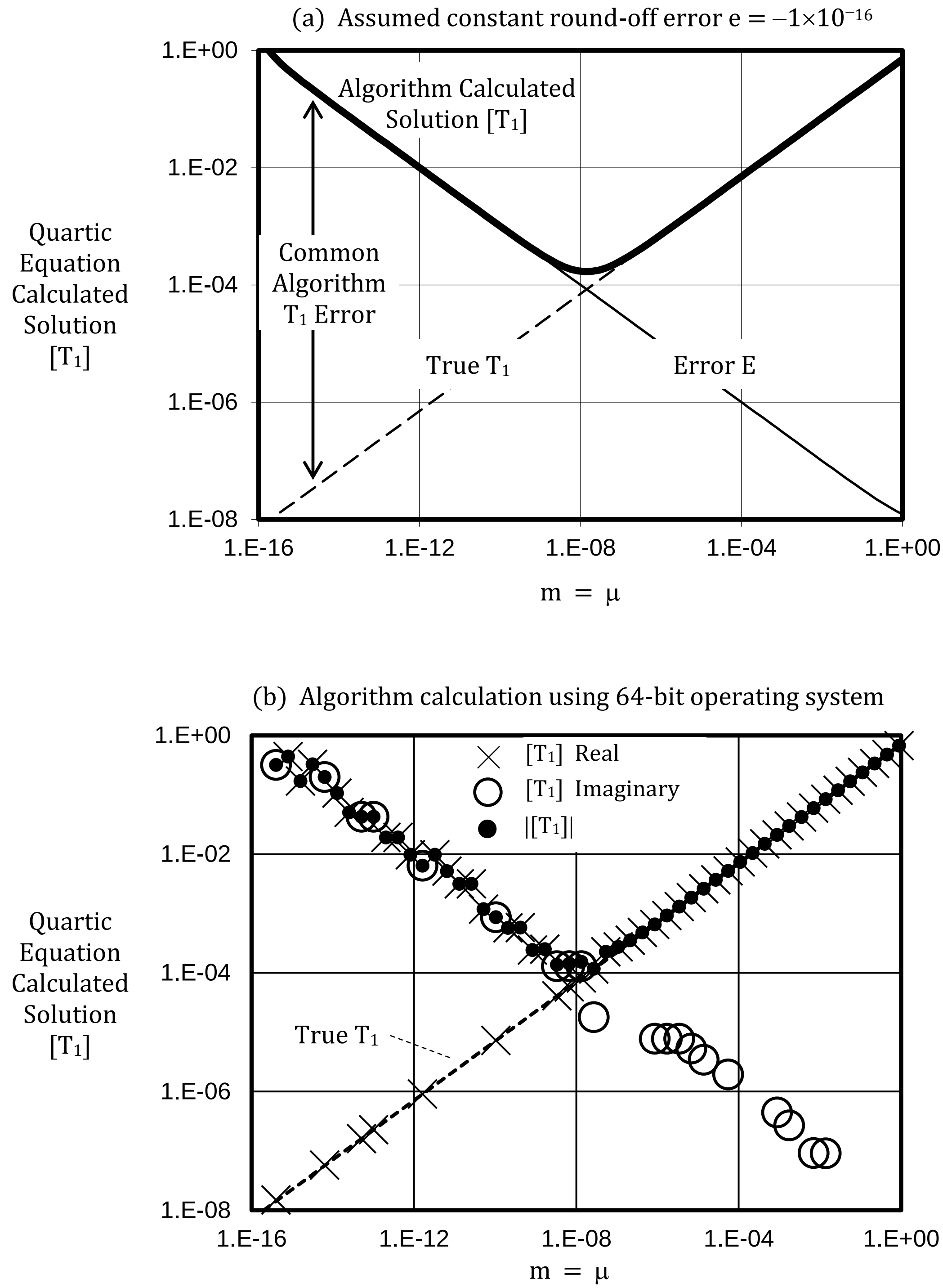
Substitute (A-3), $\mu = m$, and (A-5) into (A-7) and simplify to obtain

$$[T_1] = \sqrt{(m+e)/2} + E$$

where $E = \sqrt{2f(e/m) - e/2}$ and $f(e/m) = (1 + e/m)^{-1/2} - 1.$

We consider only the case $m = \mu > |e|$. If e is negative $(0 < -e < m)$, then $2f(e/m) - e/2$ is positive, and E is real. If e is positive, then $2f(e/m) - e/2$ is negative, and E is imaginary. If $e = 0$, then $E = 0$ and $[T_1] = \sqrt{m/2} = \sqrt{\mu/2} = T_1$.

The figure on the next page demonstrates graphically the instability of the Ferrari common algorithm as $m = \mu$ becomes small. The first graph plots E, $[T_1]$, and true T_1 versus $m = \mu$ for an assumed constant round-off error $e = -1\times10^{-16}$. This e value is typical for a 64-bit operating system applied to this problem. The plot shows how error E increases as m diminishes. Error E starts to dominate $[T_1]$ as m falls to 10^{-8}. This m value is the square root of $|e|$ and 10^8 times as great as $|e|$. As m decreases further and approaches 10^{-16}, E and $[T_1]$ increase without limit whereas the true T_1 value approaches $10^{-8}/\sqrt{2}$.



Sample Depressed Quartic Equation: $T_n^4 + (4-\mu)T_n^2 - 8\sqrt{\mu/2}\,T_n + 2\mu + \mu^2/4 = 0$

Figure A-1 Computational Instability of Ferrari Common Algorithm for Small m

The second graph demonstrates the same effect by using a 64-bit operating system to calculate b_2, b_1, and b_0 from μ, then calculate [m] as the solution of resolvent cubic equation (A-6), and finally calculate [T_1] as the solution of (A-7). The graph plots [T_1] as its real part, its imaginary part, and its modulus. True $T_1 = \sqrt{\mu/2}$ is also plotted for reference.

In summary, the fraction $b_1/(2\sqrt{2m})$ in the Ferrari common T_n formulas produces computational instability. The value m, calculated as the solution of the resolvent cubic equation, typically contains a small round-off error e not found in b_1. As m diminishes to the magnitude of e, then the calculated fraction $[b_1/(2\sqrt{2m})]$ is dominated by error. The instability is particularly severe in the case illustrated in Figure A-1 above. The calculated value [T_1] suffers large error even when the m value is several orders of magnitude greater than the round-off error e. If error e is negative, then the error in [T_1] can become unbounded as m approaches the modulus of e. By reason of this instability, the Ferrari common algorithm is not recommended for general calculation.