6.252 NONLINEAR PROGRAMMING

LECTURE 8

OPTIMIZATION OVER A CONVEX SET;

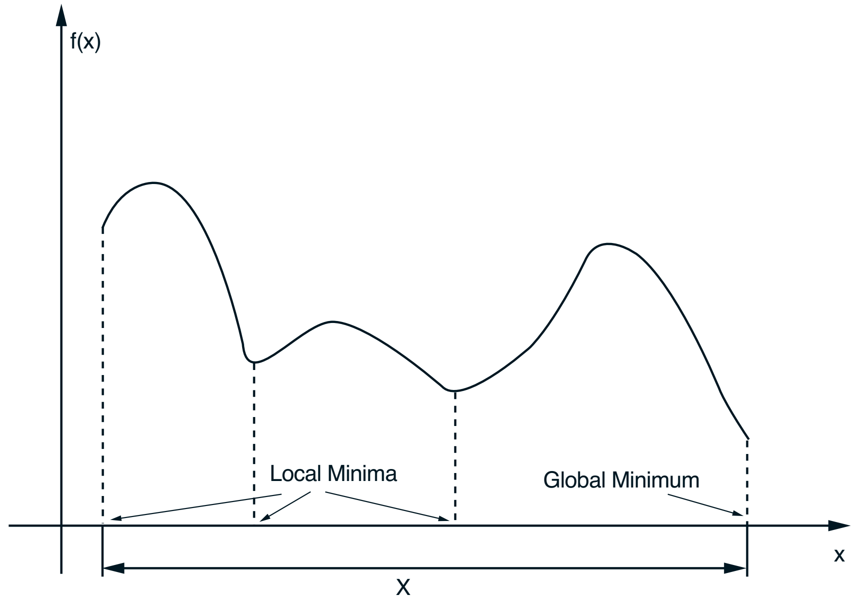
OPTIMALITY CONDITIONS

Problem: $\min_{x \in X} f(x)$, where:

(a) $X \subset \Re^n$ is nonempty, convex, and closed.

(b) f is continuously differentiable over X.

• Local and global minima. If f is convex local minima are also global.



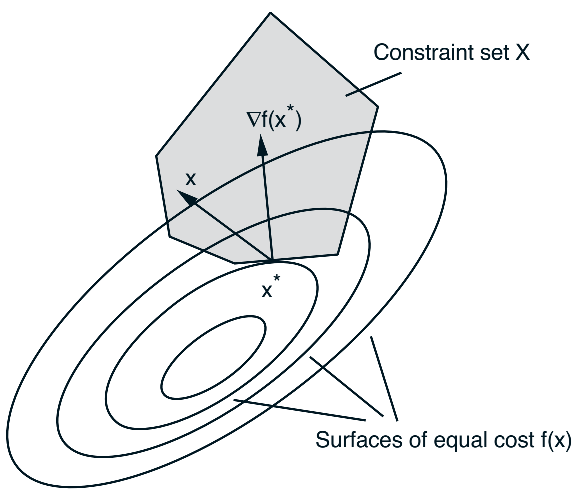
OPTIMALITY CONDITION

Proposition (Optimality Condition)

(a) If x^* is a local minimum of f over X, then

$$\nabla f(x^*)'(x - x^*) \geq 0, \qquad \forall\, x \in X.$$

(b) If f is convex over X, then this condition is also sufficient for x^* to minimize f over X.



At a local minimum x^*, the gradient $\nabla f(x^*)$ makes an angle less than or equal to 90 degrees with all feasible variations $x - x^*$, $x \in X$.

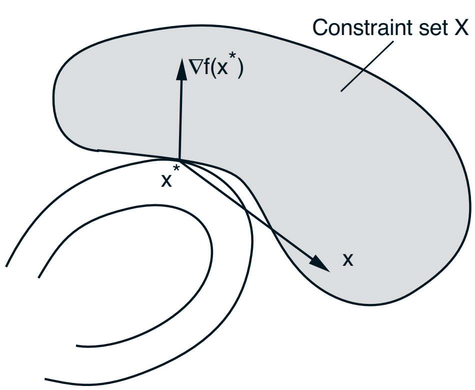


Illustration of failure of the optimality condition when X is not convex. Here x^* is a local min but we have $\nabla f(x^*)'(x - x^*) < 0$ for the feasible vector x shown.

PROOF

Proof: (a) By contradiction. Suppose that $\nabla f(x^*)'(x - x^*) < 0$ for some $x \in X$. By the Mean Value Theorem, for every $\epsilon > 0$ there exists an $s \in [0,1]$ such that

$$f\big(x^* + \epsilon(x - x^*)\big) = f(x^*) + \epsilon \nabla f\big(x^* + s\epsilon(x - x^*)\big)'(x - x^*).$$

Since ∇f is continuous, for suff. small $\epsilon > 0$,

$$\nabla f\big(x^* + s\epsilon(x - x^*)\big)'(x - x^*) < 0$$

so that $f\big(x^* + \epsilon(x - x^*)\big) < f(x^*)$. The vector $x^* + \epsilon(x - x^*)$ is feasible for all $\epsilon \in [0,1]$ because X is convex, so the optimality of x^* is contradicted.

(b) Using the convexity of f

$$f(x) \geq f(x^*) + \nabla f(x^*)'(x - x^*)$$

for every $x \in X$. If the condition $\nabla f(x^*)'(x - x^*) \geq 0$ holds for all $x \in X$, we obtain $f(x) \geq f(x^*)$, so x^* minimizes f over X. **Q.E.D.**

OPTIMIZATION SUBJECT TO BOUNDS

- Let $X = \{x \mid x \geq 0\}$. Then the necessary condition for $x^* = (x_1^*, \ldots, x_n^*)$ to be a local min is

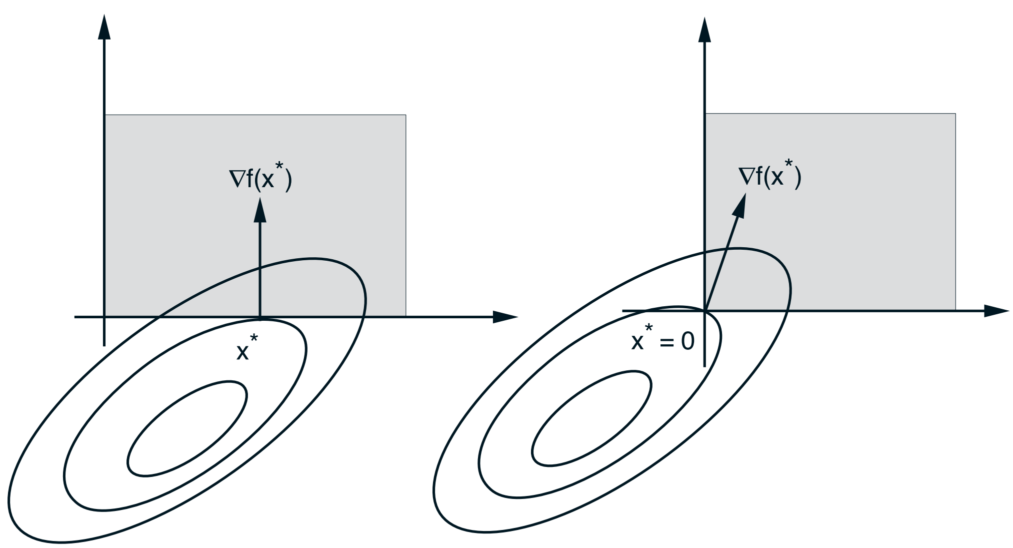
$$\sum_{i=1}^{n} \frac{\partial f(x^*)}{\partial x_i}(x_i - x_i^*) \geq 0, \qquad \forall\, x_i \geq 0,\ i = 1, \ldots, n.$$

- Fix i. Let $x_j = x_j^*$ for $j \neq i$ and $x_i = x_i^* + 1$:

$$\frac{\partial f(x^*)}{\partial x_i} \geq 0, \qquad \forall\, i.$$

- If $x_i^* > 0$, let also $x_j = x_j^*$ for $j \neq i$ and $x_i = \frac{1}{2}x_i^*$. Then $\partial f(x^*)/\partial x_i \leq 0$, so

$$\frac{\partial f(x^*)}{\partial x_i} = 0, \qquad \text{if } x_i^* > 0.$$



OPTIMIZATION OVER A SIMPLEX

$$X = \left\{ x \;\middle|\; x \geq 0, \sum_{i=1}^{n} x_i = r \right\}$$

where $r > 0$ is a given scalar.

- Necessary condition for $x^* = (x_1^*, \ldots, x_n^*)$ to be a local min:

$$\sum_{i=1}^{n} \frac{\partial f(x^*)}{\partial x_i}(x_i - x_i^*) \geq 0, \qquad \forall\, x_i \geq 0 \text{ with } \sum_{i=1}^{n} x_i = r.$$

- Fix i with $x_i^* > 0$ and let j be any other index. Use x with $x_i = 0$, $x_j = x_j^* + x_i^*$, and $x_m = x_m^*$ for all $m \neq i, j$:

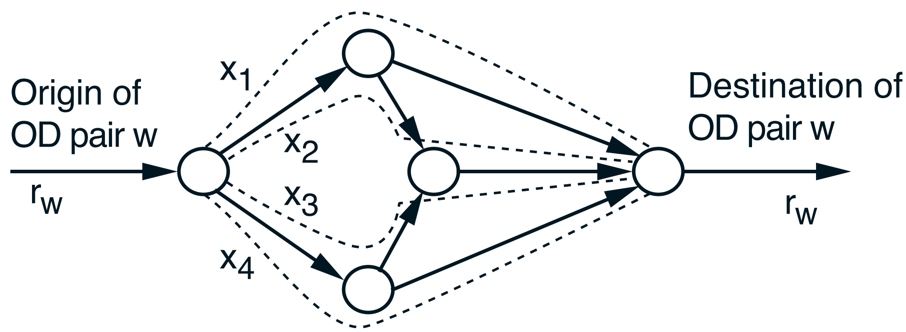
$$\left(\frac{\partial f(x^*)}{\partial x_j} - \frac{\partial f(x^*)}{\partial x_i} \right) x_i^* \geq 0,$$

$$x_i^* > 0 \quad \Longrightarrow \quad \frac{\partial f(x^*)}{\partial x_i} \leq \frac{\partial f(x^*)}{\partial x_j}, \qquad \forall\, j,$$

i.e., at the optimum, positive components have minimal (and equal) first cost derivative.

OPTIMAL ROUTING

- Given a data net, and a set W of OD pairs $w = (i, j)$. Each OD pair w has input traffic r_w.



- Optimal routing problem:

$$\text{minimize} \quad D(x) = \sum_{(i,j)} D_{ij} \left(\sum_{\substack{\text{all paths } p \\ \text{containing } (i,j)}} x_p \right)$$

$$\text{subject to} \quad \sum_{p \in P_w} x_p = r_w, \quad \forall\, w \in W,$$

$$x_p \geq 0, \quad \forall\, p \in P_w,\ w \in W$$

- Optimality condition

$$x_p^* > 0 \quad \Longrightarrow \quad \frac{\partial D(x^*)}{\partial x_p} \leq \frac{\partial D(x^*)}{\partial x_{p'}}, \qquad \forall\, p' \in P_w,$$

i.e., paths carrying > 0 flow are shortest with respect to first cost derivative.

TRAFFIC ASSIGNMENT

- Transportation network with OD pairs w. Each w has paths $p \in P_w$ and traffic r_w. Let x_p be the flow of path p and let $T_{ij}\left(\sum_{p:\text{ crossing }(i,j)} x_p\right)$ be the travel time of link (i,j).

- **User-optimization principle:** Traffic equilibrium is established when each user of the network chooses, among all available paths, a path of minimum travel time, i.e., for all $w \in W$ and paths $p \in P_w$,

$$x_p^* > 0 \quad \Longrightarrow \quad t_p(x^*) \le t_{p'}(x^*), \qquad \forall\, p' \in P_w,\ \forall\, w \in W$$

where $t_p(x)$, is the travel time of path p

$$t_p(x) = \sum_{\substack{\text{all arcs } (i,j) \\ \text{on path } p}} T_{ij}(F_{ij}), \qquad \forall\ p \in P_w,\ \forall\ w \in W.$$

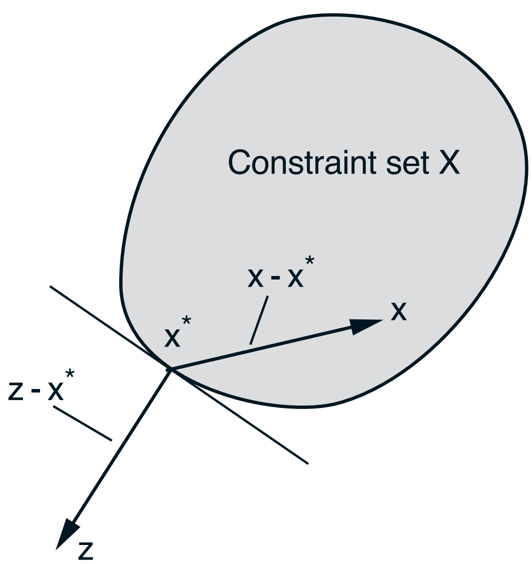
Identical with the optimality condition of the routing problem if we identify the arc travel time $T_{ij}(F_{ij})$ with the cost derivative $D'_{ij}(F_{ij})$.

PROJECTION OVER A CONVEX SET

• Let $z \in \Re^n$ and a closed convex set X be given. Problem:

$$\text{minimize} \quad f(x) = \|z - x\|^2$$
$$\text{subject to} \quad x \in X.$$

Proposition (Projection Theorem) Problem has a unique solution $[z]^+$ (the projection of z).



Necessary and sufficient condition for x^* to be the projection. The angle between $z - x^*$ and $x - x^*$ should be greater or equal to 90 degrees for all $x \in X$, or $(z - x^*)'(x - x^*) \leq 0$

• If X is a subspace, $z - x^* \perp X$.

• The mapping $f : \Re^n \mapsto X$ defined by $f(x) = [x]^+$ is continuous and nonexpansive, that is,

$$\|[x]^+ - [y]^+\| \leq \|x - y\|, \qquad \forall\, x, y \in \Re^n.$$