FIGURE 6.38 Search for the local center on each one of the five links of the network of Figure 6.37.

Consider, for instance, the link (A, B) with a length of 4 units. For all points x on (A, B) we can find and plot the functions $d(x, j)$ for $j = A, B, C, D, E$. For instance, if we set, by convention, $x = 0$ at A and $x = 4$ at B, we have

$$d(x, B) = 4 - x \qquad 0 \leq x \leq 4$$

and

$$d(x, D) = \begin{cases} 1 + x & 0 \leq x \leq 3.5 \\ 4 + (4 - x) & 3.5 \leq x \leq 4 \end{cases}$$

The functions $d(x, A)$, $d(x, B)$, $d(x, C)$, $d(x, D)$, and $d(x, E)$ are all shown on Figure 6.38a for the link (A, B). Now, since, by definition,

$$m(x) = \text{Max}\ (d(x, A), d(x, B), d(x, C), d(x, D), d(x, E))$$

the function $m(x)$ is given by the upper envelope for the five functions as shown on Figure 6.38a. Obviously, the local center of link (A, B) is at a point 0.5 unit away from A (and 3.5 units away from B) and $m(x_\ell) = 3.5$.

Repeating the same procedure for the other four links, we finally obtain

link (A, B): local center 0.5 unit from A; $m(x_\ell) = 3.5$

link (A, D): local center at A and at D; $m(x_\ell) = 4$

link (B, C): local center at C; $m(x_\ell) = 3$

link (D, E): local center at D; $m(x_\ell) = 4$

link (C, D): local center 0.5 unit from C; $m(x_\ell) = 2.5$

This completes Step 1 of the algorithm. In Step 2 we choose the point x^* on the link (C, D) and 0.5 unit away from C for the location of the absolute center; $m(x^*) = 2.5$.

From our example we conclude that:

1. The absolute center and the vertex center do not have to coincide. In fact, the absolute center *does not* have to be on a link emanating from the node where the vertex center is located—as was the case in our example.

2. The maximum distance function, $m(x)$, is piecewise linear and its slope is always $+1$ or -1.

From the second remark above, the following result can be easily derived [ODON 74]:

Theorem: For the local center, x_ℓ, on a link (p, q),

$$m(x_\ell) \geq \frac{m(p) + m(q) - \ell(p, q)}{2} \qquad (6.26)$$

where, as usual, $\ell(p, q)$ denotes the length of link (p, q).

Exercise 6.7 Prove the validity of the theorem.

From this theorem and from the observation that, by definition, $m(i^*) \geq m(x^*)$ (i.e., the maximum distance associated with the vertex center must be greater than or equal to the corresponding distance for the absolute center), we can derive the following simple test:

If for a link (p, q),

$$\frac{m(p) + m(q) - \ell(p, q)}{2} \geq m(i^*) \qquad (6.27)$$

then the local center x_ℓ of (p, q) cannot improve on $m(i^*)$ (and, therefore, need not be found).

This test, taking advantage of the fact that it is very simple to find $m(i^*)$, often leads to considerable reduction in the computation effort required to obtain the absolute center (see also Problem 6.12).

Example 17 (continued)

With respect to our five-node, five-link example, we found easily that the vertex center is at node C and that $m(i^*) = m(C) = 3$.

Applying our test to the five links of the graph, we then obtain

link (A, B): $\quad \dfrac{m(A) + m(B) - \ell(A, B)}{2} = \dfrac{4 + 5 - 4}{2} = 2.5 \ (< 3)$

link (A, D): $\quad \dfrac{m(A) + m(D) - \ell(A, D)}{2} = \dfrac{4 + 4 - 1}{2} = 3.5 \ (> 3)$

link (B, C): $\quad \dfrac{m(B) + m(C) - \ell(B, C)}{2} = \dfrac{5 + 3 - 2}{2} = 3 \ (= 3)$

link (D, E): $\quad \dfrac{m(D) + m(E) - \ell(D, E)}{2} = \dfrac{4 + 5 - 1}{2} = 4 \ (> 3)$

link (C, D): $\quad \dfrac{m(C) + m(D) - \ell(C, D)}{2} = \dfrac{3 + 4 - 2}{2} = 2.5 \ (< 3)$

Therefore, the local center need be found only for links (A, B) and (C, D)—a significant savings in computational effort.

We also note that a highly efficient algorithm exists for finding the absolute center when the network at hand happens to be a tree (see Problem 6.11). The algorithm [HAND 73] is the following:

Single-Tree-Center Algorithm (Algorithm 6.13)

Let G be a tree network and let e_i $(i = 1, 2, \ldots, m)$ represent the end vertices (i.e., the nodes of degree 1) of the tree. Then:

STEP 1: Choose arbitrarily any point $x \in G$ and find the (end) vertex, say e_s, farthest away from x.

STEP 2: Find the (end) vertex, say e_t, which is farthest away from e_s.

STEP 3: The absolute center of G is at the midpoint of the path from e_s to e_t. The vertex center of G is at the node that is closest to the absolute center.

This last algorithm does not even require computing the minimum distance matrix for the tree network G!

6.5.5 Multiple Centers

By analogy to the k-medians problem, there is also a k-centers problem. The following definitions are appropriate.

Let $G(N, A)$ be an undirected network and let $X_k = \{x_1, x_2, \ldots, x_k\}$ be a set of k points on G. We shall use, as before, $d(X_k, j) = \text{Min}_{x_i \in X_k} d(x_i, j)$ [i.e., $d(X_k, j)$ is the minimum distance between any one of the points $x_i \in X_k$ and the node j on G].

Definition: A set of k points X_k^* on G is a set of *unconstrained* (or *absolute*) k-centers of G, if for every set $X_k \in G$,

$$m(X_k^*) \leq m(X_k) \qquad (6.28)$$

where

$$m(X_k) = \underset{j \in N}{\text{Max}}\, d(X_k, j) \qquad (6.29)$$

Definition: If the sets X_k, X_k^* in Definition 1 are constrained to consist solely of k nodes of the node set N, then the set X_k^* is a set of *vertex k-centers* of G.

Until recently, k-center problems were believed to be among the most difficult graph problems to solve. The work of Handler [HAND 79] has, however, provided a set of algorithms—which he called "relaxation algorithms"—that solve efficiently problems of considerable size (e.g., $n = 200$, $k = 5$).

6.5.6 Requirements Problems

So far we have addressed urban facility location problems of the type: "Where should I locate k facilities to maximize (or minimize) some (given) objective function?" Very often, however, the question will be asked in quite different terms: "We would like to achieve certain standards of performance (either as specified by legislative fiat or as deemed necessary by service administrators). What is then the smallest (or least costly) number of facilities

that we need, and where should these facilities be located to achieve these standards?"

In this section we shall discuss briefly procedures for dealing with this second type of question—which we shall refer to as a "requirements problem."

Clearly, our earlier work (and algorithms) can provide the building blocks for solving requirements problems. To take a concrete example, the Emergency Medical Service Systems Act passed by Congress in 1973 (EMSS Act PL93-154) states in its guidelines that 95 percent of rural calls for emergency medical service should be reached within 30 minutes from the call and 95 percent of urban calls within 10 minutes. This is now a case where some standards of performance have been preset by legislation for an urban (and rural) service. At this point the analyst must take over. To determine appropriate locations for basing the emergency medical care facilities or an associated ambulance system, the foregoing specifications must be interpreted in more concrete operational terms. For instance, the following might be a reasonable interpretation of the standards of performance set by the EMSS Act: "It is required that 95 percent of all calls must be reached within 30 (10) minutes. We also know that, in most reasonable service systems, it might be expected that a certain percentage of calls for service will have to queue up for a period of time due to all the servers of the service system being busy. It might then be inferred that for the EMS system to have any hope of achieving the specified performance standards, it must be that all of the potential users of the service should be within 30 (or 10) minutes of travel time from their closest EMS facility."

We thus now require a set of locations such that no potential users are more than 30 (or 10) minutes away from at least one of them. This we recognize as a problem very similar to the k-centers problem. In this case, however, the number of required locations, k, is not given. Instead, we know the maximum acceptable distance that can be associated with our k-centers. In other words, in our notation, we are given the value of $m(X_k^*)$ ($= 30$ or 10 minutes, depending on the case), and we are asked to find k and the locations X_k^*.

A possible approach to finding the least number and the locations of EMS facilities required to achieve $m(X_k^*) = 30$ (or 10) should now be obvious.[18]

EMS Coverage "Algorithm"

STEP 1:   Set $k = 1$.

STEP 2:   Solve the k-centers problem for the current value of k.

[18]We are now assuming that link lengths are given in terms of travel time rather than distance.

STEP 3:   If $m(X_k^*) \leq 30$ minutes (or 10 minutes), *stop*. The current value of k is the minimum number of facilities required and the appropriate locations of these facilities are the locations of the k-centers. If $m(X_k^*) > 30$ (or 10) minutes, go to Step 4.

STEP 4:   Set $k = k + 1$ and return to Step 2.

In the procedure above we did not specify whether at Step 2 we are solving an absolute k-centers problem or a vertex k-centers problem. This will depend on whether the potential locations of the facilities are unrestricted or, as is so often the case in practice, the choice of locations is restricted to only a finite number of points (or general areas) in the region of interest. In the latter case the vertex k-centers problem is the appropriate one for Step 2.

We can now describe in quite general terms a more realistic "scenario" than hitherto, for facility location problems in the urban environment.

1. A certain *primary objective* is stated, usually in the form of a requirement for compliance with the performance standards set by an administrative, legislative, or other body (e.g., "achieve the performance standards set by the EMSS Act").

2. Some restrictions on the potentially acceptable facility locations are also specified. These restrictions are often due to local considerations or conditions or to special requirements of the facilities to be constructed (e.g., "all ambulance stations must be adjacent to local hospitals").

3. One or more secondary objectives are also often specified. These secondary objectives are usually expressed in terms of costs, although other measures of effectiveness also appear sometimes (e.g., "once the performance standards in the primary objective are met, the least-cost system configuration should be selected").

Example 18: Ambulance Location/Allocation in a Rural Area

In a case reported recently [JARV 75a], a team of analysts were asked to assist the Bel-O-Mar Regional Council of Wheeling, West Virginia, in developing a regional emergency ambulance allocation plan. Bel-O-Mar is a regional planning agency responsible for a four-county region encompassing the Wheeling SMSA.[19] This is a predominantly rural area of about 1,360 square miles straddling the Ohio River; its population of 200,000 people is dispersed among approximately 90 communities in the area.

The primary objective was to design a system that complies with the EMSS Act standards described earlier. The secondary objective—once the primary one was satisfied—was to minimize average response time, a fact

[19]Standard Metropolitan Statistical Area.

that finally resulted in ambulances being allocated to the seven communities which were the largest demand-generators in the region. Finally, local considerations included the following: ambulances should not cross the Ohio River in responding to calls; ambulances must be located in both Wheeling and Bethlehem—the two largest communities in the area; and ambulances allocated to certain prespecified communities cannot provide service outside these communities.

Similar examples have been reported for fire departments, for sanitation departments, and for emergency facilities in general. It should also be emphasized that the secondary objective(s) often plays an important role in the determination of facility locations. The reason is that in many problems one finds numerous combinations of locations that achieve the primary objective of satisfying the preset performance standards. The ties must then be resolved with reference to the secondary objective(s). The secondary objective(s) usually calls for the solution of a k-medians type of problem, since, as we have stated, it is usually concerned with the minimization of some average-cost function.[20]

6.5.7 Set-Covering Problems

Another subproblem that often arises in connection with solving requirements problems is known as the set-covering problem. It can be described as follows.

Consider a set $Y_n = \{y_1, y_2, \ldots, y_n\}$ of n points on a network G (for instance, Y_n could be the set of demand-generating nodes on G) and another set $X_m = \{x_1, x_2, \ldots, x_m\}$ of m points on G which are candidates for the location of a set of facilities (some of the points in X_m may coincide with points in Y_n). Let us now assume that an unambiguous threshold of performance (e.g., a maximum distance λ) has been specified so that any location $x_j \in X_m$ can be viewed as either satisfying or falling short of achieving that level of performance with respect to any point $y_i \in Y_n$. Then we say that a point $x_j \in X_m$ "covers" ("does not cover") a point $y_i \in Y_n$ if the point x_j satisfies (does not satisfy) the threshold of performance with respect to point y_i. For instance, when the threshold of performance is a maximum distance λ, then x_j covers y_i if $d(x_j, y_i) \leq \lambda$ and does not cover it if $d(x_j, y_i) > \lambda$. The similarity with the concepts of coverage in Section 3.6 is obvious.

The set-covering problem then consists of finding the minimum number, say k^*, of points from the set X_m such that all points in Y_n are covered.

[20]In many cases it is true that primary and secondary objectives are the reverse of what was stated above. That is, the primary objective is to minimize an average-cost function and the secondary one to achieve a given level of performance.

Example 19: Set-Covering Problem

Consider seven urban locations A through G and five distinct points V through Z in the same urban area. All 12 points lie on an urban road network. We shall call the points A through G the "demand set" of this problem and points V through Z the "facilities set." We wish to find the smallest number of points in the facilities set needed to cover the demand set of points if it is specified that each point in the demand set must be within 20 minutes of travel distance from a point in the facilities set. The 5×7 matrix of minimum distances (in minutes) on the network is given in Table 6-11.

TABLE 6-11 Minimum distance matrix for Example 19.

$$[d(i,j)] = \begin{array}{c|ccccccc} & A & B & C & D & E & F & G \\ \hline V & 3 & 9 & 13 & 22 & 18 & 24 & 21 \\ W & 25 & 21 & 12 & 7 & 36 & 27 & 12 \\ X & 12 & 24 & 27 & 8 & 24 & 16 & 26 \\ Y & 9 & 4 & 9 & 17 & 26 & 22 & 14 \\ Z & 21 & 29 & 28 & 30 & 10 & 31 & 28 \end{array}$$

The minimum distance matrix $[d(i,j)]$ can immediately be translated into the *coverage* matrix, $[c(i,j)]$, by setting each matrix element $c(i,j)$ to

$$c(i,j) = \begin{cases} 1, & \text{if } d(i,j) \leq 20 \\ 0, & \text{otherwise} \end{cases}$$

TABLE 6-12 Coverage matrix for Example 19 ($\lambda = 20$).

$$[c(i,j)] = \begin{array}{c|ccccccc} & A & B & C & D & E & F & G \\ \hline V & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ W & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ X & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ Y & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ Z & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

The coverage matrix for our problem is shown in Table 6-12. Potential facility point W, for example, can cover demand points C, D, and G.

The set-covering problem (SCP) can now be stated simply as follows. Reduce the coverage matrix $[c(i,j)]$ to the minimum number of rows required so that each column in the reduced matrix has at least a single element equal to 1. The problem in this form can be readily formulated as an integer programming 0–1 problem, and indeed the best-known approaches to it

[GARF 72, TORE 71] begin with such a formulation and develop general algorithms for the solution of SCP's.

Here we shall not describe these general algorithms. We shall outline instead a very simple matrix-reduction algorithm which may be used to reduce the computational size of SCP's. In many cases, especially those involving facility location problems on urban or rural roadway networks, this matrix-reduction algorithm simplifies SCP's so much that a solution can be obtained by inspection upon completion of the reduction process.

Matrix Reduction for Set Covering (Algorithm 6.14)

STEP 1 (*Feasibility Check*):   If there is at least one column in the coverage matrix that consists entirely of zeroes, *stop*. No feasible solution exists (i.e., the performance standards for coverage must be relaxed or more points must be added to the facilities set).

STEP 2:   If any columns have only one nonzero element, say in row i^*, then the point corresponding to row i^* must receive a facility. Include that point in the list of those that must receive a facility and eliminate row i^* and all columns having a 1 in row i^* from the matrix.

STEP 3:   If any row(s) i'' is such that all its entries are less than or equal to the corresponding entries of another row i' [i.e., if $c(i'', j) \leq c(i', j)$ for all j], then eliminate row i''.

STEP 4:   If any column(s) j'' is such that all its entries are greater than or equal to the corresponding entries of another column j' [i.e., if $c(i, j'') \geq c(i, j')$ for all i], then eliminate column j''.

STEP 5:   Repeat Steps 2–4 until either (a) the coverage matrix becomes completely empty or (b) no columns or rows are eliminated during a complete pass through Steps 2–4. In case (a), a complete solution (minimum number of facilities and their locations) has been obtained on termination. In case (b), a solution may be obtainable by inspection on termination, or application of a more sophisticated SCP algorithm to the reduced matrix may be necessary.

The rationale for each one of Steps 1–5 is rather obvious. The following example may also be helpful in understanding Algorithm 6.14.

Example 19 (continued)

Let us apply Algorithm 6.14 to the set-covering problem already described. The initial coverage matrix $[c(i, j)]$ is given in Table 6-12. By inspection we can

determine that a feasible solution does exist, since all columns of $[c(i, j)]$ contain at least a single nonzero entry (Step 1).

In Step 2, we find that column F has only one nonzero element at row X. It follows that a facility *must* be located at X to serve, at the least, demand point F. We eliminate from the coverage matrix row X and columns A, D, and F [since $c(X, A) = c(X, D) = c(X, F) = 1$] and obtain the reduced matrix shown in Table 6-13a.

TABLE 6-13 Various steps in matrix reduction for set covering (Example 19).

$$
\begin{array}{c}
\begin{array}{cccc} B & C & E & G \end{array} \\
\begin{array}{c} V \\ W \\ Y \\ Z \end{array}
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$
(a)

$$
\begin{array}{c}
\begin{array}{cccc} B & C & E & G \end{array} \\
\begin{array}{c} V \\ Y \end{array}
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}
\end{array}
$$
(b)

$$
\begin{array}{c}
\begin{array}{cc} E & G \end{array} \\
\begin{array}{c} V \\ Y \end{array}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\end{array}
$$
(c)

At Step 3, rows W and Z are eliminated (Table 6-13b). All entries of row W are less than or equal to the corresponding entries of row Y, and therefore all points covered by a facility at W are also covered by a facility at Y. The same is true with respect to rows Z and V, respectively. Next, at Step 4, columns B and C are eliminated since their entries are all greater than or equal to the corresponding entries of column E (or, for that matter, G)— hence any facility location that covers demand point E will also cover demand points B and C. The reduced matrix on completion of Step 4 is shown in Table 6-13c. Finally, on return to Step 2, facilities must be located at both points V and Y, and the algorithm terminates since upon elimination of these two rows, the coverage matrix is empty.

Thus, the minimum set of locations consists of points V, X, and Y. We may now, if we wish, return to the original minimum distance matrix $[d(i, j)]$ (see Table 6-11) and assign each demand point to its nearest facility. This would assure, in addition to coverage within less than 20 minutes, minimization of average travel distance, as well. In this way, we assign demands from A and E to V, from D and F to X, and from B, C, and G to Y.

6.5.8 Related Problems

The three classes of facility location problems on networks that we have reviewed here (medians, centers, and requirements problems) are not only important in themselves but also illustrate the basic techniques for approaching a wide variety of related classes of problems. Two examples of such problems follow.

The first is concerned with locating so-called "obnoxious" facilities that everyone wishes to be as far away from as possible. Garbage incinerators are one example of this type of facility. Airports (to a much lesser extent) are another. One instance of such a problem is called the *maxian* problem and is concerned with finding a set X_k^* of locations on a network so as to *maximize* the function

$$J(X_k) = \sum_{j=1}^{n} h_j \, d(X_k, j)$$

Thus, this problem differs from the median problem only in the fact that it is concerned with maximizing rather than minimizing $J(X_k)$ [CHUR 78]. This, however, requires a very different solution approach than in the case of medians.

Our second example is *cent-dian* problems [HALP 78]. Rather than have a primary and a secondary objective like the requirements problems, these problems combine a minisum and a minimax objective into a single weighted average; that is, they seek to minimize

$$T(X_k) = \lambda J(X_k) + (1 - \lambda)m(X_k)$$

where λ is a constant $(0 < \lambda < 1)$ and $J(X_k)$ and $m(X_k)$ are as defined in (6.15) and (6.29), respectively. Many municipal service facilities (e.g., recreation centers, basketball and tennis courts, "little city halls") can be viewed as cent-dians: they should not be too far from any segment of the population *and* they should maximize accessibility to the *average* citizen.

There is also an almost endless list of variations of requirements (and set-covering) problems, many of which have been motivated by urban applications [TORE 71, GARF 72].

6.6 PROBABILISTIC NETWORKS

In all the network models of urban and other areas that we have seen so far, we have not considered the possibly probabilistic nature of at least some of the networks' parameters. Yet, in many instances this probabilistic nature is sufficiently important to merit separate attention.

For instance, at no point in the discussion above was explicit recognition given to the fact that demand for urban services constitutes, in most cases, a Poisson (or some other probabilistic) process. A consequence of this is that these services undergo periods of intensive activity, as well as other periods of relative inactivity. During high activity periods, any given facility or, more specifically, the servers associated with that facility may be unable to keep up with the many demands for service. In such instances, either some demands will have to queue up and wait for a server to become available or assistance may be sought from other service facilities elsewhere. In either event, we have a violation of some of the unstated premises under which all the *facility location problems* described earlier were examined. For, in the k-center, the k-median, and the set-covering problems, we have implicitly assumed that:

1. No queueing ever occurs (i.e., whenever a demand arises at a point assigned to a given facility, that facility will immediately respond to that demand).

2. Facilities do not interact (i.e., once a demand generation point is assigned to a given facility that point is served forever and exclusively by that facility with no assistance from servers located at other facilities).

When violation of these assumptions is an infrequent occurrence, as it usually is in service systems that are utilized at a relatively low rate, the results obtained in Section 6.5 with regard to optimality of locations can still be considered valid. If, however, as it happens with highly utilized urban service systems, the foregoing assumptions are violated with high probability, then these results must be accordingly modified and revised. To do this, we need the tools provided by queueing theory—and especially the methodology of the hypercube queueing model—which we covered in Chapters 4 and 5. Much recent work [JARV 75b, BERM 78] has focused on the problem of selecting facility locations on networks in the presence of significant queueing (see also Problems 6.13 and, especially, 6.17 for some important concepts in this respect).

Rather than examine the implications of the probabilistic aspects of demand our emphasis in this section will be on another important network parameter, "link length." In our earlier material we have always assumed that the lengths of the links are known constants. In practice, however, this is often not so, especially when these lengths represent travel *time* rather than *distance* between adjacent nodes. As we all know from experience, there is usually considerable uncertainty about how long it will take to travel between any two points x and y in a city. This is especially true under peak traffic

conditions, when actual travel times can vary widely from day to day over any given travel path.

It may thus be much more realistic to represent link lengths as random variables (hopefully, with known probability distributions) under these circumstances. We shall use the term *probabilistic network* to refer to all networks for which one or more link lengths are explicitly defined to be random variables.[21]

6.6.1   Description and Some Properties of Probabilistic Networks

It is possible to identify at least three factors that contribute to random variations in travel times on urban roadway networks: (1) random fluctuations in traffic density, traffic lights, and so on, and the attendant variations in travel speeds that result in what could be termed "routine" randomness; (2) changes in the average value of the volume of traffic during hourly, weekly, and seasonal cycles creating "hour-of-the-day" variations in travel times; and (3) accidents, changes in weather conditions, and other unpredictable events causing "nonroutine" randomness.

Depending on the context, each one (or combinations) of the randomness-inducing factors above could be the focus of attention in an application. For instance, since decisions on locations of immovable facilities (e.g., a hospital or a firehouse) are of the "strategic" type (i.e., concerned with making a good location choice in the long-term sense), it is variations of the second kind above that one would be particularly concerned with in such cases. That is, we would be mostly interested in the probability distribution of travel times over the course of an average day. This distribution, in turn, can be used as the data base needed to optimize in some sense the accessibility of the facility(ies) over the facility's lifetime.

By contrast, when a city is making contingency plans for, say, a major snowstorm, it is randomness of the third kind that is of interest. The planner must estimate accessibility measures on the basis of the likely probability distributions for travel times after the snowstorm (e.g., impassable side streets, traffic jams on main thoroughfares, etc.).

No matter what the case is, the representation of (some or all) link lengths as random variables usually means an enormous increase in the computational effort required to answer some of the problems that we solved rather easily earlier for deterministic networks. The main reason is that the shortest distance between any two points on a graph and the associated path are much more difficult to compute or even to conceptualize when link lengths are random variables. Since the computation of shortest distances

[21]Our earlier, deterministic models can of course be viewed as only special cases of probabilistic networks.

plays such a fundamental role in most network-based problems, this difficulty extends to the solution of these other problems as well.

To see why computation of shortest distances is such a mathematically cumbersome procedure on probabilistic networks, consider the case of an undirected graph $G(N, A)$ and the problem of finding the shortest path between some given pair of nodes $s, t \in N$. Assume further that there are q alternative paths, $\pi_1, \pi_2, \ldots, \pi_q$, between s and t, not all of which are necessarily disjoint (see also Figure 6.39). Suppose that these paths include a total of k links—with some links possibly belonging to more than one path and that the length of link i is a random variable L_i ($i = 1, 2, \ldots, k$). Then the joint probability density function for the link lengths is given by $f_{L_1, L_2, \ldots, L_k}(\ell_1, \ell_2, \ldots, \ell_k)$. [In the case where the link lengths are statistically independent random variables, $f_{L_1, L_2, \ldots, L_k}(\ell_1, \ell_2, \ldots, \ell_k) = \prod_{i=1}^{k} f_{L_i}(\ell_i)$; that is, the joint pdf is equal to the product of the individual pdf's for each link length].
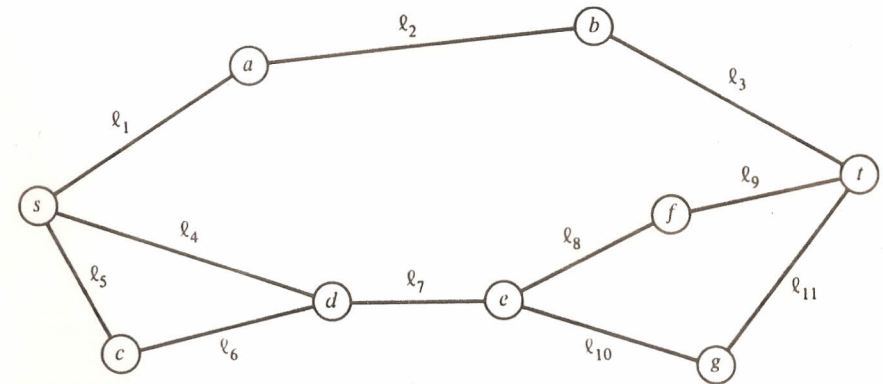

FIGURE 6.39   A graph with 11 links, and 5 alternative paths leading from *s* to *t* (e.g. (*s, a, b, t*), (*s, d, e, f, t*), etc.)

If we now define

$$X_j = \text{the length of the path } \pi_j = \sum_{\text{all links} \in \pi_j} L_i \qquad (6.30)$$

we have

$$D(s, t) = \text{shortest distance between nodes } s \text{ and } t = \text{Min}(X_1, X_2, \ldots, X_q) \qquad (6.31)$$

Computing the probability distribution for the shortest distance between s and t is thus equivalent to computing the probability density function for the minimum of the q random variables in (6.31). To obtain the probability distribution for $D(s, t)$, $F_{D(s,t)}(d)$, the following steps are therefore necessary:

STEP 1: From the joint density function $f_{L_1, L_2, \ldots, L_k}(\ell_1, \ell_2, \ldots, \ell_k)$ and from the set of q equations of type (6.30) for the path lengths X_J, obtain the joint probability density function for the path lengths

$$f_{X_1, X_2, \ldots, X_q}(x_1, x_2, \ldots, x_q).$$

STEP 2: Use the relationship

$$F_{D(s, t)}(d) = 1 - \int_d^\infty \cdots \int_d^\infty f_{X_1, X_2, \ldots, X_q}(x_1, x_2, \ldots x_q) \, dx_1 \cdot dx_2 \cdots \cdot dx_q$$

to compute the quantity of interest, since $D(s, t)$ is defined as in (6.31).

Obviously, for even a small number of paths, π_J, and a small number of member links, m, the two steps above (especially the first) will involve very tedious and time-consuming mathematical manipulations and operations. This is truly unfortunate, as this is one of the most commonly faced problems in the dispatching of urban emergency vehicles, where choosing the shortest—in some probabilistic sense—path is often of paramount importance. Only a few general results with reference to the above two-step procedure have been derived [FRAN 69], but they are of very limited practical use.

A more mathematically tractable problem is the one concerned with pairwise comparisons of distinct paths. This case arises whenever, in dispatching a vehicle, it is necessary to choose between exactly two given alternative routes on the roadway network. In such cases we can compare the two routes not only with respect to their expected values (which is a trivial matter) but also in a probabilistic sense [FRAN 69]. For example, we may be concerned with guaranteeing that extremely long travel times will be avoided as often as possible.

Given two alternative paths (routes) π_1 and π_2, we shall say that path π_1 is "d_0 *better than*" π_2 if

$$P\{X_1 \geq d_0\} < P\{X_2 \geq d_0\} \tag{6.32}$$

where X_1 and X_2 are the lengths of π_1 and π_2, respectively, and d_0 is a constant.

When link lengths are statistically independent random variables, the pdf's for X_1 and X_2 can be computed through a sequence of convolutions involving the pdf's of the link lengths for all the links in each one of the two paths (see also Chapter 2). If, however, the paths π_1 and π_2 contain many links, then it is natural to use a normal approximation for the distributions of X_1 and X_2, taking advantage of the fact that, under very general condi-

tions, the sums of random variables are approximately normally distributed (Central Limit Theorems). Note that it is possible for π_1 to be d_1 better than π_2 *and* for π_2 to be d_2 better than π_1 for two constants d_1 and d_2 ($d_1 \neq d_2$).

6.6.2 Discrete and Finite State Space for Probabilistic Networks

As a consequence of the conceptual and computational difficulties in obtaining shortest distances on probabilistic networks we are usually forced to make some simplifications when working with such networks. One obvious (and by far the most common) possible simplification is to use the expected link length for every link as a proxy for the true link length. This of course transforms what is actually a probabilistic network into a deterministic network with constant link lengths.

Another possible simplification which, while increasing mathematical tractability, preserves at the same time the added realism provided by the probabilistic properties of the models is the assumption that link lengths are discrete random variables that take only a finite number of values. That is, we approximate the probability distribution of random variable $L(i, j)$, the length of the link (i, j), by writing

$$P[L(i, j) = \ell_s] = p_s, \qquad \text{for } s = 1, 2, \ldots, c \tag{6.33}$$

where $\sum_{s=1}^c p_s = 1$. Thus, the link length can only take values $\ell_1, \ell_2, \ldots, \ell_c$. If one of these values, say ℓ_k, is infinite, we say that link (i, j) "fails with probability p_k" since, in this case, it will be impossible with probability p_k to traverse (i, j) in finite time.

The net effect of this assumption is that the network now has only a finite number, m, of *states*. Each state differs from the others by a change in the length of at least one link. Thus, the finest-grained sample space for the network consists of a listing of the set of all m mutually exclusive and collectively exhaustive possible network states that can be denoted as $G_1, G_2, \ldots, G_m$ (to make explicit the fact that each state is a different "snapshot" of the network) with each network state having a probability of occurrence P_r, $r = 1, 2, \ldots, m$.

In general, the number of states will depend on the degree of statistical dependence among the random variables $L(i, j)$. In the extreme case where complete statistical independence prevails, we have

$$m = \prod_{\text{all links}(i, j)} c_{ij}$$

where c_{ij} is the number of values that the length of link (i, j) [i.e., $L(i, j)$] can take. Similarly, the state probabilities, P_r, must either be known [in the case

when the $L(i, j)$ are not mutually independent] or are easily computable (when we have statistical independence).

Let us now define:

$\ell_r(i, j) =$ length of link (i, j) when the network is in state G_r

$d_r(x, y) =$ shortest distance between points x and y when the network is in state G_r

For each given state, r, the network's snapshot G_r looks exactly like a deterministic network and, therefore, using our earlier shortest-path algorithms and the known link lengths $\ell_r(i, j)$, we can compute the minimum distance $d_r(x, y)$ between any two points $x, y \in G$.

In principle, then, it is possible to compute, under the assumption that there is no change of states during travel, such quantities as:

1. The length of the expected shortest path between two points $x, y \in G$:

$$E[D(x, y)] = \sum_{r=1}^{m} P_r d_r(x, y) \qquad (6.34)$$

2. The probability that the distance between two points $x, y \in G$ is less than or equal to a constant λ:

$$P[D(x, y) \leq \lambda] = \sum_{r \in S} P_r \qquad (6.35)$$

where S is the set of states for which $d_r(x, y) \leq \lambda$.

The computational effort involved may, naturally, still be formidable and depends critically on the total number m of distinct states. For instance, supposing that each link length $L(i, j)$ takes c distinct values and assuming full statistical independence between links, we have

$$m = c^e \qquad (6.36)$$

where e is the total number of links in the network. Since for a connected graph we have $n - 1 \leq e \leq n(n - 1)/2$, where n is the number of nodes in G, the number of states in the case above can be very large even for modest-size networks.

On the other hand, it is very unlikely that in an urban context the link lengths will be statistically independent: traffic conditions, for example, have approximately identical time patterns throughout a city, and therefore at peak traffic times the higher values of link lengths prevail simultaneously, for all links on the network model, while the opposite is true at off-peak periods. Thus, in many cases it is possible that a few distinct states (small m) will

suffice to model, at least approximately, the most common sets of travel conditions under which urban roadway networks operate. The approach that was outlined above will then prove computationally feasible under these circumstances.

Example 20: PMF for the Shortest Path Length

Consider again a graph like the one of Figure 6.40, in which all but three of the links are deterministic. The lengths of the probabilistic links $L(b, t)$, $L(s, d)$, and $L(e, t)$ are assumed to take two, three, and two distinct values, respectively, with probabilities as indicated. Thus, for instance, $P\{L(b, t) = 6\} = \frac{2}{3}$. We assume statistical independence between links and wish to find the distribution of the shortest distance $D(s, t)$ between nodes s and t.

There are a total of 12 states in this case. They have been indexed with the numbers 1 through 12 (in arbitrary order) and their important parameters are tabulated in Table 6-14. Figure 6.41 shows the network in the state which
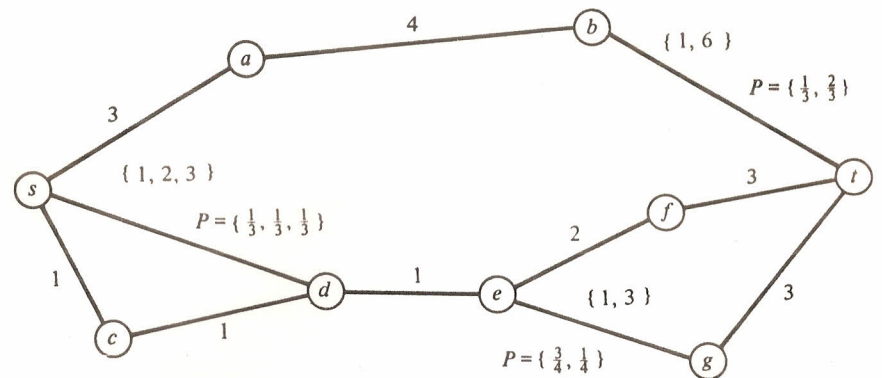

FIGURE 6.40 The network of Figure 6.39 with links (b, t), (s, d) and (e, g) having probabilistic lengths. Lengths and corresponding probabilities (when applicable) are shown on each link.
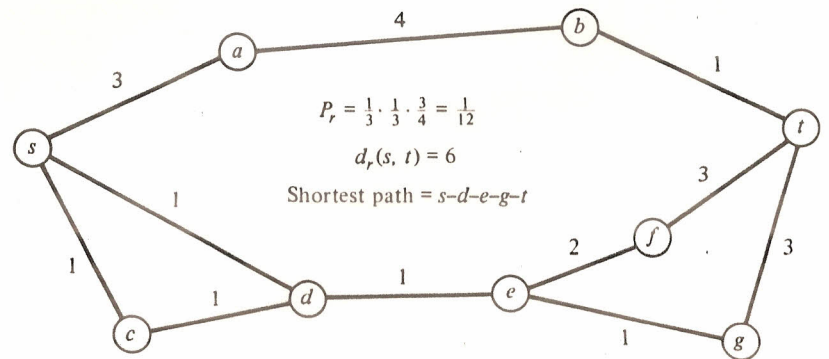

FIGURE 6.41 One particular state of the network of Figure 6.4.

has been indexed as state 1 in Table 6-14. From Table 6-14 it can be seen that $D(s, t)$ takes the values 6, 7, and 8 with probability 1/4, 7/12, and 1/6, respectively.

TABLE 6-14   Parameters for Example 20.

State r	(b, t)	(s, d)	(e, g)	P_r	Shortest Path[1]	$d_r(s, t)$
1	1	1	1	$\frac{1}{12}$	$\{s, d, e, g, t\}$	6
2	1	1	3	$\frac{1}{36}$	$\{s, d, e, f, t\}$	7
3	1	2	1	$\frac{1}{12}$	$\{s, d, e, g, t\}$	7
4	1	2	3	$\frac{1}{36}$	$\{s, a, b, t\}$	8
5	1	3	1	$\frac{1}{12}$	$\{s, c, d, e, g, t\}$	7
6	1	3	3	$\frac{1}{36}$	$\{s, a, b, t\}^*$	8
7	6	1	1	$\frac{1}{6}$	$\{s, d, e, g, t\}$	6
8	6	1	3	$\frac{1}{18}$	$\{s, d, e, f, t\}$	7
9	6	2	1	$\frac{1}{6}$	$\{s, d, e, g, t\}^*$	7
10	6	2	3	$\frac{1}{18}$	$\{s, d, e, f, t\}^*$	8
11	6	3	1	$\frac{1}{6}$	$\{s, c, d, e, g, t\}$	7
12	6	3	3	$\frac{1}{18}$	$\{s, c, d, e, f, t\}$	8

[1]An * indicates that there is at least one additional equal-length shortest path at that network state.

6.6.3   Facility Locations on Probabilistic Networks

As a further illustration of the application of probabilistic network ideas to urban service system problems, we now reexamine briefly the question of facility location assuming a probabilistic network model. Throughout our discussion we shall be using the discrete and finite state-space model which we described in Section 6.6.2: each probabilistic network can be in one of m states with the probability of it being in state r given by P_r.

Two important observations must be made in the beginning:

1. When there are two or more facilities on a probabilistic network and assuming that any demand point will always be served by its nearest facility at the moment when that demand is generated, the facility serving a particular demand point will depend on the state of the network. This point becomes quite obvious once it is realized that shortest distances and paths change with the state of the network.

2. A finest-grained sample space should be used in solving facility location problems on probabilistic networks. This can best be explained initially through the following example.

Example 21: *Using the Finest-Grained Sample Space*

Consider the network of Figure 6.42. The lengths $\ell(1, 3)$, $\ell(2, 3)$, $\ell(2, 4)$, and $\ell(3, 4)$ are deterministic with values of 5, 2, 5, and 5 units, respectively, while $L(1, 2)$ takes on the values of 1 or 13 units, with probability 0.5 for each value. Thus, $E[L(1, 2)] = 7$.

Assume now that we wish to solve a single-median problem and that somehow it is known that the optimal location for the median is at one of the nodes of the network.[22] The demand weights are shown in parentheses next to each node.
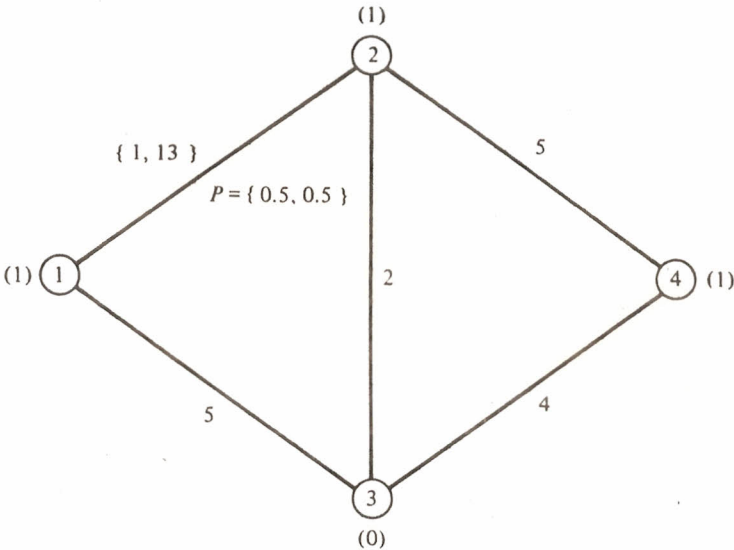

FIGURE 6.42   Probabilistic network for Example 21.

Suppose that $E[L(1, 2)]$ were substituted for the length $L(1, 2)$. Then the optimal location of the facility is at node 3 with an expected travel time of $3\frac{2}{3}$ units $(= \frac{1}{3} \cdot 5 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 4)$. If, however, we use a finest-grained sample space and assuming that the current state of the network is always known whenever a trip is undertaken, it can be seen that the minimum distance between nodes 1 and 2 is either 1 or 7 units [via node 3 when $L(1, 2) = 13$], each with probability 0.5. The expected travel distance when the facility is located at node 2 is then equal to 3 units $[= \frac{1}{3}(4) + \frac{1}{3}(0) + \frac{1}{3}(5)]$ and node 2 is a better location than node 3.

From the example it is clear that the finest-grained sample space is needed because the shortest path between any two points may vary with the state

[22]We have shown this to be true for deterministic networks, but we have yet to prove it for probabilistic networks.

of the network. In fact, our example also demonstrated that the substitution of expected value $E[L(i,j)]$ for the actual values $\ell_r(i,j)$ of the link lengths may lead to erroneous results. Unfortunately, as we noted earlier, this substitution is often used in practice, since most deterministic network models are constructed, in the first place, by using "average" travel times as deterministic substitutes for the random variables $L(i,j)$ that link lengths really are. Such a substitution may be justified only when the current state of the network is *not* known whenever a trip is started.

We now proceed to develop a single-median result for probabilistic networks. For an undirected network $G(N,A)$ with n nodes, we have:

Definition: A point x^* on an undirected probabilistic network G is an *expected 1-median* of G, if for every point $x \in G$,

$$\bar{J}(x^*) \leq \bar{J}(x) \tag{6.37}$$

where

$$\bar{J}(x) = \sum_{r=1}^{m} \sum_{j=1}^{n} P_r h_j d_r(x,j) \tag{6.38}$$

Our result, completely analogous to Hakimi's for deterministic networks, states:

Theorem: At least one solution to the expected 1-median problem exists on a node in an undirected probabilistic network.

This theorem can be proved (see [MIRC 79b]) under the following assumption:

Homogeneity assumption: The time required to travel a fraction θ of the link (i,j) is equal to $\theta \cdot \ell_r(i,j)$ for all $r = 1, 2, \ldots, m$.

The homogeneity assumption states in effect that the speed of travel on any given link is uniform—an assumption that is both straightforward and reasonable since the network model of the urban area can easily be constructed such that this assumption holds.

The theorem above can be extended, as one might suspect, to the case of the *expected k-medians*. In fact, the following even more general result has been shown to hold [MIRC 79b]:

Definition: A set of k points X_k^* on an undirected probabilistic network G is a set of *expected optimal k-medians* of G if for every set $X_k \in G$,

$$\bar{J}_u(X_k^*) \geq \bar{J}_u(X_k) \tag{6.39}$$

where

$$\bar{J}_u(X_k) = \sum_{r=1}^{m} \sum_{j=1}^{n} P_r h_j u(d_r(X_k,j)) \tag{6.40}$$

and $u(t)$ is the utility function of travel time t.

Then

Theorem: At least one set of expected optimal k-medians exists on the nodes in an undirected stochastic network if the utility function for travel time is convex.

Similar very general results for medians also apply to directed networks.

Finally, we note that an implicit assumption in the foregoing discussion of the medians problem on probabilistic networks has been that the appropriate connectivity conditions apply. For instance, our definition of the expected single median would be meaningless if it were not true that "those nodes with nonzero demand ($h_j > 0$) are always, that is, for all m possible states of the network, connected." Were this not the case, then some distance(s) between one or more nodes and the facility would be infinite and so would the expression for $\bar{J}(x)$.

The study of probabilistic networks[23] and their applications to urban service systems has begun in earnest only recently. This is not due to lack of promise of highly useful results but rather to the conceptual and computational difficulties to which we alluded earlier. It can be safely anticipated that this will be one of the most active areas of investigation by operations researchers, management scientists, and others in the next few years.

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Problems

6.1 *Shortest paths with some $\ell(i,j) < 0$*   When some link lengths on a network are negative the two shortest-path algorithms of Section 6.2 must be modified. Algorithm 6.1 must be modified drastically, while only a minor modification is necessary in Algorithm 6.2. In this problem we shall explore various aspects of the shortest-path problem with some $\ell(i,j) < 0$.

a.   Let $\ell(f,e) = -2$, instead of 5, in the graph of Figure 6.3. How would the shortest-path tree of Figure 6.5 change?

b. Suppose that the following approach has been suggested for finding shortest paths on the graph of Figure 6.3 with $\ell(f, e) = -2$: (i) Add 2 units to the length of all links of the graph so that all $\ell(i, j) \geq 0$; (ii) use Algorithm 6.1 to find any shortest paths required and then subtract 2 units from every link on each shortest path to find the true length of each shortest path. What is wrong with such an approach?

c. Suppose now that $\ell(f, e) = -8$ in Figure 6.3. What would now be the minimum distance between nodes a and j, $d(a, j)$?

    Hint: Be careful!

d. The phenomenon that you have observed in part (c) is referred to as a "negative cycle." Whenever a negative cycle exists between two nodes of a graph, the shortest-path problem for this pair of nodes is meaningless. Note that this means that no *undirected* links on a graph should have negative $\ell(i, j)$—since this immediately implies a negative cycle. Shortest-path algorithms must be able to "detect" the presence of negative cycles if they are to work with some $\ell(i, j) < 0$. The key to such detecting is the following statement: in a graph with n nodes, no meaningful shortest path (i.e., one that does not include a negative cycle) can consist of more than $n - 1$ links. Argue for the validity of this statement.

e. Algorithm 6.2 can be used as stated for cases where some $\ell(i, j) < 0$ with only a minor modification to check for the existence of negative cycles upon termination. How would you use the final matrix $D^{(n)}$ at the conclusion of Algorithm 6.2 to check whether there are any negative cycles in a graph?

    Hint: What should happen to one or more diagonal elements $d_n(i, i)$ of this matrix if there is a negative cycle in the graph?

f. Repeat Example 2 of Section 6.2.2 for the case in which the length of the directed arc from node 5 to node 2 in Figure 6.6 is equal to -3.

g. Can you suggest how shortest-path Algorithm 6.1 should be changed in order to be applicable to cases with some $\ell(i, j) < 0$?

    Hints: No labels can become permanent (i.e., nodes cannot become closed) until all labels are permanent; the algorithm requires at *most* $n - 1$ passes but may terminate earlier if no labels change during a pass.

    For a more extensive discussion of algorithms of this type, see, for instance, Chapter 8, Section 2.2, of Christofides [CHRI 75].

6.2 *Intersection of two-way streets*   Consider the intersection of two major avenues near the center of a city. The intersection is controlled by a set of traffic lights. Both avenues carry two-way traffic.

Cars wishing to make a right turn at the intersection incur a "penalty" of two

time units; those making a left turn, a penalty of four time units; and those continuing on a straight course, a penalty of one time unit. No U-turns are permitted.

Draw a network model of this intersection. The model should be a finest-grained one, so that all possible actions of motorists at this intersection can be accounted for.

6.3 *Network model of commuter's choices*   In the case shown in Figure P6.3, a commuter wishes to travel from his residence near station A to his place of employment at D. The commuter's transportation choices are:

1. Ride a bus from A to D; ride time is 25 minutes.

2. Ride on subway line 1 to station B, change to subway line 2 and move to D. Ride times for each leg are shown in Figure P6.3.

3. Ride on subway line 1 to station C, change to subway line 3 and move to D. Ride times for each leg are shown in Figure P6.3.
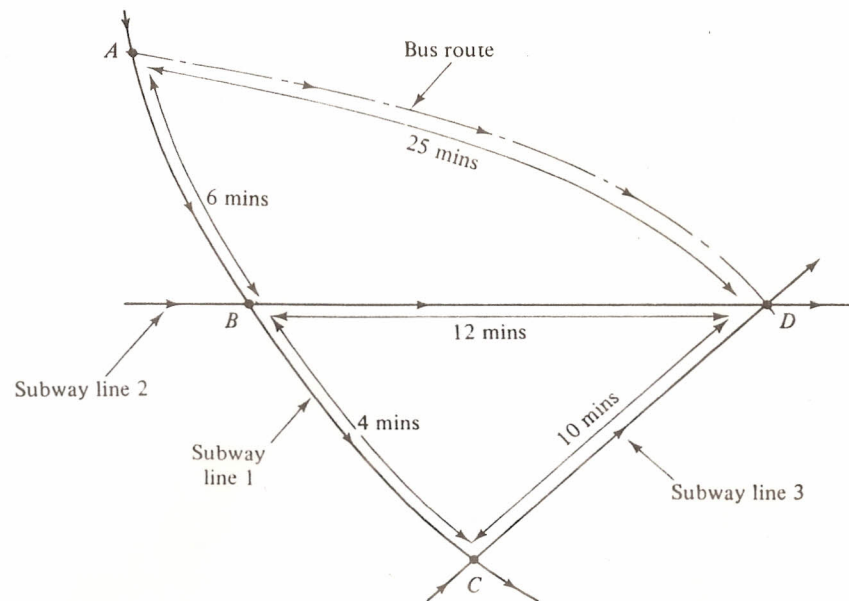
Headways between subways on all lines are exactly 10 minutes. Moreover, the subway schedules are coordinated so that a line 2 train to D passes through B exactly 4 minutes after a line 1 train has stopped at B, and similarly, a line 3 train to D passes through C exactly 4 minutes after a line 1 train has stopped at C. Assume that transfer times and stop time are negligible.

Headways between buses at A are also constant and equal to 10 minutes.

a. Assuming that the commuter is aware of the bus schedule and of the subway schedule and that he times his arrival at A so that it coincides with the passage of whatever vehicle he chooses to ride, prepare a network model

for the situation shown in Figure P6.3. The network model should be such that a shortest-path algorithm, such as Algorithm 6.1, can be used to find the shortest path from *A* to *D*.

b. Assume now that all headways in this problem (for buses and subways) have negative exponential pdf's all with means of 10 minutes. The commuter now arrives at a random time relative to the passage of buses or subways from *A*. In addition, all subway lines are operating independently of each other or of the bus. Prepare a network model for this new situation such that one can determine, through a shortest-path algorithm, the transportation mode and route that offers the shortest expected travel time between *A* and *D*.

c. What is the preferred route (and mode) in each of parts (a) and (b)? What is the shortest expected travel time in each case?

6.4 *Design of an optimum road network* Suppose that in Figure 6.11, the nodes of the graph represent seven towns in a rural area and its links a set of paved roads which *could possibly* be constructed to connect the towns. Note that some connections (e.g., *C* to *D*) cannot be built (owing, perhaps, to such constraints as mountainous terrain). The distances indicated on Figure 6.11 are miles.

Suppose now that a regional commission charged with planning the road network in this area: (1) has a budget sufficient to construct up to a total of 34 miles of paved roads; and (2) wishes to minimize the quantity $Z = \sum_i \sum_j d(i, j)$, $i, j = A, B, C, D, E, F, G$, where $d(i, j)$ is the shortest distance between town *i* and *j* measured on the roadway network that will actually be constructed. [If there is no paved path between two towns *i* and *j*, $d(i, j)$ is considered infinite. Note that the MST provides the minimum budget for which *Z* is finite.]

Find the optimum road network for this case. (This is an example of "optimum network design," a class of difficult network problems.)

6.5 *Solving the CPP on a city map* It has been pointed out that good solutions to urban Chinese postman problems can usually be found quite easily, even for large networks, if a good map is available.

The map of Figure P6.5 involves about 50 nodes and 90 arcs. It is based on a test problem due to P. Authier (University of Sherbrooke, Canada). There are 38 odd-degree nodes in the network (i.e., more than 10^{21} alternative matchings)! Solve the CPP for this network using Algorithm 6.5 and finding a matching of odd-degree nodes by inspection (plus trial and error). Make a photocopy of the map and indicate (by doubling lines in red pencil or in ink) the street segments that must be traversed twice in your solution. (A tour is required; distances are in tens of feet.)

What is the total distance that will be traversed twice in your solution? (In the optimal solution the answer is 8,360 feet, including the length of dead-end streets—which must be traversed twice anyway.)

6.6 *Chinese postman problem on a directed network* To solve the CPP on a directed network [BELT 74], we begin by quoting the version of Euler's theorem that applies to directed networks:
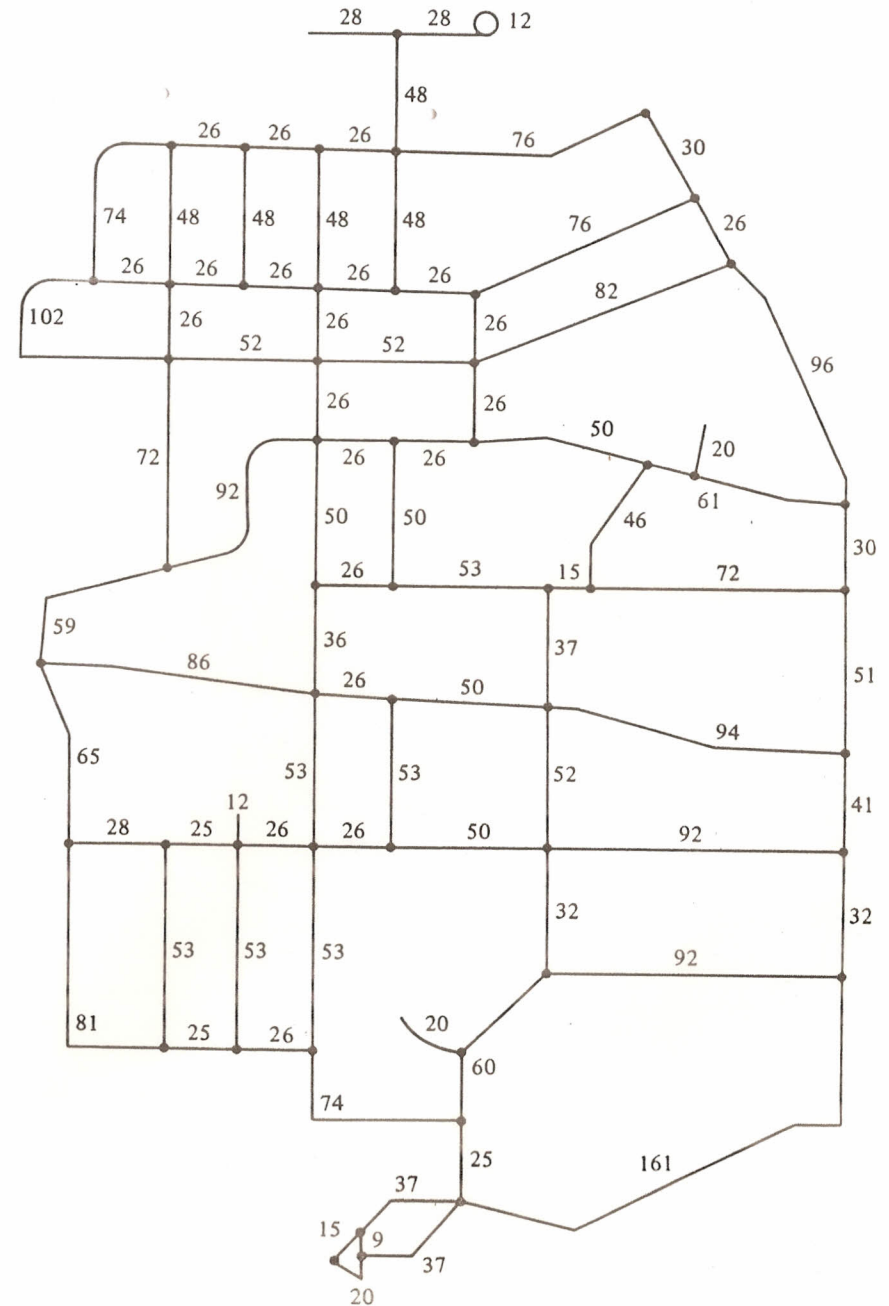
A connected directed network possesses an Euler *tour* if and only if the indegree and the outdegree of every one of its nodes are equal.

The proof of this theorem is completely analogous to the proof of Euler's theorem for undirected networks (you may wish to retrace its steps).

To solve the CPP on any directed graph, we first define

$$P_i = \text{“polarity” of } i = (\text{indegree of } i) - (\text{outdegree of } i)$$

A node i for which $P_i > 0$ $(P_i < 0)$ is called a "supply" ("demand") node. We indicate the sets of supply and demand nodes as S and D, respectively.

a.   Show that $\sum_{i \in N} P_i = 0$.

The following algorithm solves the CPP on directed graphs (the algorithm is stated informally):

STEP 1:   Identify all supply and demand nodes and compute the polarities of each and the minimum distance $d(i, j)$ from all nodes $i \in S$ to all nodes $j \in D$.

STEP 2:   Solve a transportation problem (TP) to find the optimum "matchings" of supply with demand nodes. This TP is:

minimize

$$\sum_{i \in S} \sum_{j \in D} d(i, j) x_{ij}$$

subject to

$$\sum_{j \in D} x_{ij} = P_i \quad \text{for all } i \in S$$

$$\sum_{i \in S} x_{ij} = -P_j \quad \text{for all } j \in D$$

where $x_{ij} \geq 0$. (Remember that $P_j < 0$ for $j \in D$; i.e., $-P_j$ is positive.)

STEP 3:   For each $x_{ij} > 0$ in the solution to the TP, add to G, x_{ij} replications of the shortest path from $i \in S$ to $j \in D$. The resulting network G' has $P_i = 0$ for all nodes.

STEP 4:   Find an Euler tour on G'. This tour is a solution to the CPP on G.

b.   Write a paragraph explaining what the algorithm above does and why. Can any links be traversed *more than twice* in the CPP solution?

c.   Apply the algorithm above to solve the CPP on the directed network of Figure P6.6. Describe a minimum-length tour that begins and concludes at node b.

   Hint: The optimal solution requires coverage of 28 units of distance over and above the length of the network.
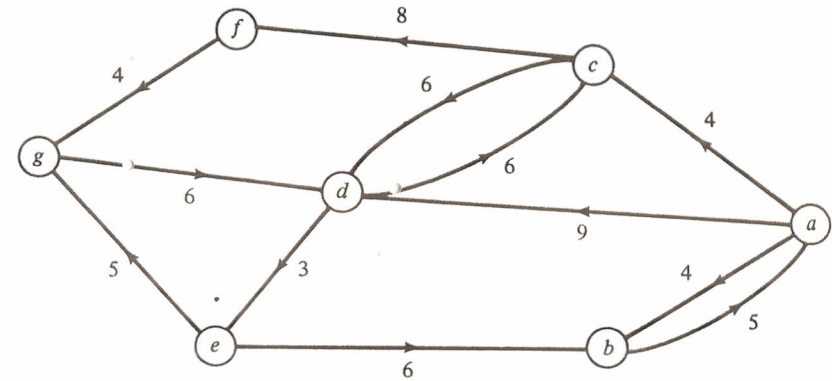
d.   Suppose now that G is a mixed graph (i.e., it has both directed and undirected links). It might be thought that in order to solve the CPP on such a network one could (1) substitute all undirected links (i, j) with two directed links (i, j) and (j, i) of equal length (and of opposite directivities); and (2) apply the algorithm (for the CPP on directed graphs) described above to the resulting directed network. What is wrong with this approach? As we noted earlier in the chapter, no efficient algorithm is available for the CPP on mixed graphs and the problem has in fact been shown to be NP-complete.

6.7 *Upper bound on the expected length of the TST*   A total of n points are randomly and uniformly distributed within a unit square. We wish to obtain an upper bound on the expected length of the optimum traveling salesman tour, $E[\text{TST}]$, through these n points.

Suppose that we divide the unit square into m equal-width columns, as shown on Figure P6.7, where m is to be determined later. We visit all n points through the following strategy. We start from the point in the leftmost column having the largest y coordinate. We then visit the point in the same column having the next lower y coordinate, and so on. When we reach the lowest point in the first column, we next visit the lowest point in the second column and we then traverse the points in that column upward. We continue this process by visiting all columns from left to right, changing the direction of traversal at each column. To complete the tour, we join the last point to the first point with a straight line.
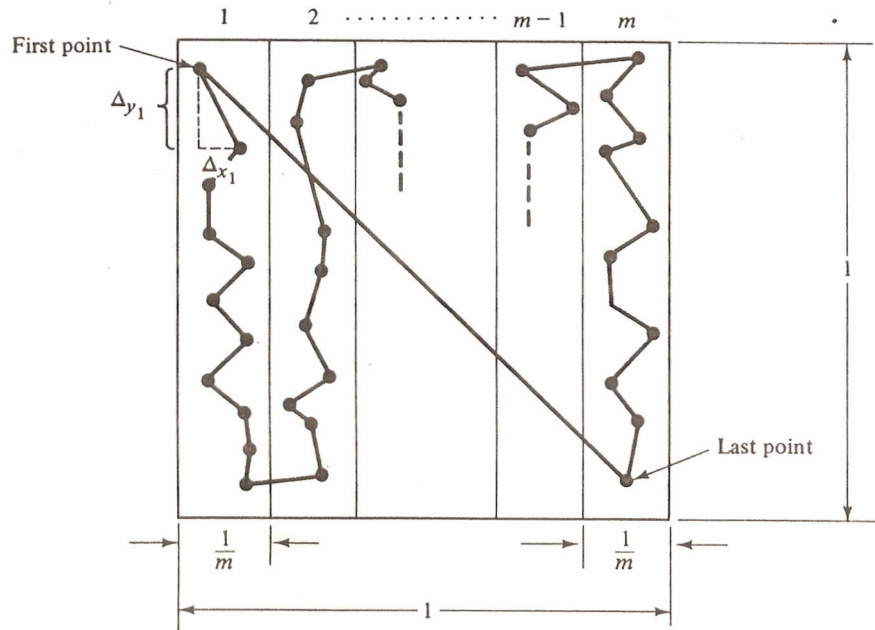
Let S_n be a random variable denoting the length of this tour. Clearly, $E[S_n] \geq E[\text{TST}]$.

a.   Let $d(P_J, P_{J+1})$ be the distance between two consecutively visited points, $P_J = (x_J, y_J)$ and $P_{J+1} = (x_{J+1}, y_{J+1})$ in S_n. It is obvious that

$$d(P_J, P_{J+1}) \leq |x_J - x_{J+1}| + |y_J - y_{J+1}| = \Delta x_J + \Delta y_J$$

Then show that

$$E[S_n] \leq (m - 1) \cdot \frac{1}{m} + (n - m)\frac{1}{3m} + m + \sqrt{2}$$

b. Now choose the most advantageous value of *m* and find an *approximate* upper bound for $E[\text{TST}]$ for *large n.* (Assume that $n \gg 3$.)

c. Suppose now that instead of a unit square, the foregoing procedure were applied to a rectangular area of dimensions X_0 by Y_0. Show from (a) and (b) that if *m* is chosen correctly (what is the proper value of *m*?) the upper bound on $E[\text{TST}]$ (i.e. $E[S_n]$) must be less than or equal to a quantity proportional to approximately $1.15\sqrt{nA}$, where $A = X_0 \cdot Y_0$, for large *n* ($n \gg 3$).

6.8 *Coin collection from parking meters* Figure P6.8 shows a section of a downtown area. A special coin-collection truck must traverse all street segments indicated with solid lines once a week, to collect coins deposited in parking meters by motorists. Parking meters exist on only one side of these streets.

Street segments indicated by dashed lines do not have parking meters and therefore *need not be traversed*, except as necessary to travel between parts of the downtown street grid.

Travel in all street segments is permitted in only one direction, as indicated by the arrows in Figure P6.8. An east-west block is 9 units long, and a north-south block is 6. Diagonal street segments are 11 units long. Note that there is no direct connection between points 3 and 7.

What is the length of the shortest route that the truck can travel beginning and ending at point 1 and traversing, at least once, all street segments with parking meters? Describe one such shortest route.
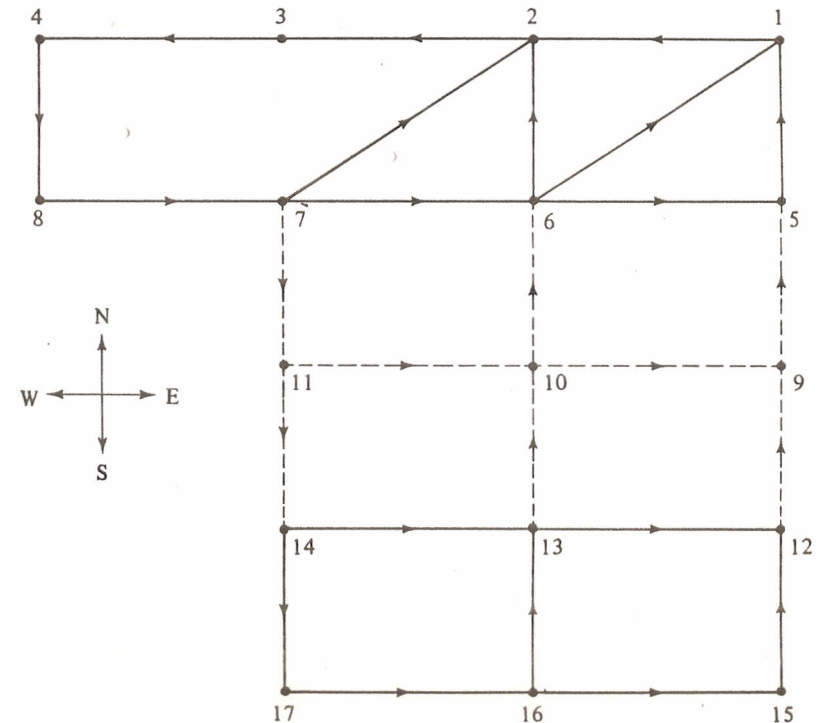
How is this problem different from the CPP on directed networks? Can you devise a systematic procedure for dealing with this family of problems?

6.9 *Proof of Hakimi's theorem* Prove that "at least one set of *k*-medians exist solely on the nodes of a network *G*" (Section 6.5.2). The proof for $k = 1$ was given in Section 6.5.2.

6.10 *Location of a "supporting facility"* Consider the network of Figure P6.10 and imagine that the nodes represent five cities, the numbers next to the nodes the "weights" of the cities, and the numbers next to the links the mile length of roadways connecting the cities. Cars travel on the roadways at an effective speed of 30 mph. Assume that a major facility, say an airport, has been located at some point on this network. A regional planning group now wishes to install a high-speed transportation link to the airport with a single station. The high-speed vehicles will travel on the network at twice the speed of cars and their route will be the shortest route to the airport. It is assumed that travelers to the airport will choose that combination of transportation modes which minimizes their access time to the airport (ignoring transfer times).

To clarify the description above, assume that the airport is at node 2 and that the single station of the high-speed link is located at node 5. Then, the access time to the airport of travelers from node 5 is 25 minutes. However, the access time of
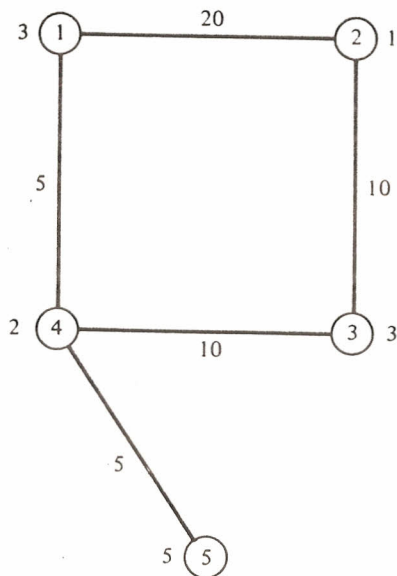
travelers from node 1 is still 40 minutes. (It would take travelers from node 1 exactly 20 minutes to get to node 5 by car and then another 25 minutes to get from 5 to 2, so that it is better to go directly to the airport by car.)

a. Show that no matter where the major facility (airport) is located, an optimal location for the station of the high-speed vehicles must be on a node of the graph. "Optimal" here means minimizing the total weighted travel distance to the airport for travelers from the five cities. Note that the airport is *not* restricted to be at one of the nodes of the network. You may wish to show this for a general network rather than for this specific case only.

b. Assuming that the airport is located at node 2, where should the station be? Note that it is simple to devise an algorithm for solving this type of problem.

c. Assume now that travel time, in minutes, by car between any two points x and y on the network is given by $f[d(x, y)] = [d(x, y)/5]^2$, where $d(x, y)$ is the shortest distance between the two points, and that travel time through the high-speed link between the same two points is given by $g[d(x, y)] = \frac{1}{2}[d(x, y)/5]^2$. How would you answer part (b) now?

Hint: The optimal solution is no longer on a node.

It can be shown that the result you proved in part (a) holds as long as the functions $f[d(x, y)]$ and $g[d(x, y)]$ are both concave in $d(x, y)$ [MIRC 79a].

6.11 *Validity of Algorithm 6.13* In this problem you are asked to prove the validity of Algorithm 6.13 for the single center of a tree. Using the notation of Section 6.5.4

and the symbol x^* to denote the (yet unknown) absolute center of the undirected tree network G:

a. Show that for all points $x \in G$ we must have

$$m(x) = m(x^*) + d(x, x^*)$$

b. Argue from part (a) that

$$m(e_s) = 2m(x^*)$$

and, therefore, that x^* must lie on the path associated with $m(e_s)$ and must be at the halfway point between e_s and e_t.

6.12 *Speeding up the search for the absolute center* In our discussion of the absolute-center problem, it was pointed out that the quantity

$$L_1(p, q) = \frac{m(p) + m(q) - \ell(p, q)}{2}$$

provides a lower bound for the value of $m(x_\ell)$ on a link (p, q) [cf. (6.26)]. This bound provides a very convenient test that facilitates the search for the absolute center of a graph.

Another bound for $m(x_\ell)$ has been developed recently, as follows. For the link (p, q), let r be the farthest node from p ($r \in N$) and s the farthest node from q ($s \in N$) [i.e., $m(p) = d(p, r)$ and $m(q) = d(q, s)$]. The quantities that will be used to develop the new bound are $d(p, s)$ and $d(q, r)$. Clearly, the quantity Max $[d(x, r), d(x, s)]$ provides a lower bound on $m(x_\ell)$.

a. Prove the following result: The quantity Max$\{d(x, r), d(x, s)\}$ may attain at most a single local minimum *within* (q, p), i.e., excluding the nodes p and q. If such a local minimum exists it is attained at a point $x_0 \in (p, q)$ which is a distance

$$\frac{\ell(p, q) + d(r, q) - d(s, p)}{2}$$

away from node p along (p, q) and at which the value of Max$\{d(x, r), d(x, s)\}$ attains the value

$$L_2(p, q) = \frac{d(p, s) + d(q, r) + \ell(p, q)}{2}$$

$L_2(p, q)$ is then a lower bound for $m(x_\ell)$ for all points $x \in (p, q)$, excluding the nodes p and q.

b. Show that $L_2(p, q) \geq L_1(p, q)$, that is, that $L_2(p, q)$ provides a better lower bound and thus a sharper test than $L_1(p, q)$ for speeding up the search for the absolute center of a graph.

6.13 *Facility location with congestion*   The network of Figure P6.13 represents five towns and the roadways connecting them. Emergency medical centers (one or more) are to be located in this area. The numbers in parentheses indicate the average number of calls for assistance generated *per hour* at each of the five towns. For each town the call-generating process can be considered Poisson and the process for each town is independent of that for all others. The numbers on the links of the network indicate the travel time, *in minutes*, for ambulances traveling that link (we assume that travel times within the towns are equal to 0).

Assume that it has been decided to locate exactly two medical facilities in this region. Each medical facility will be assigned a set of towns that it will serve exclusively (the two sets are mutually exclusive and collectively exhaustive). Once the two sets of towns have been determined, the two facilities will operate as separate and independent entities. Each facility operates in the following way.

Ambulances are stationed at each medical facility and travel to incidents (and back) along shortest paths.
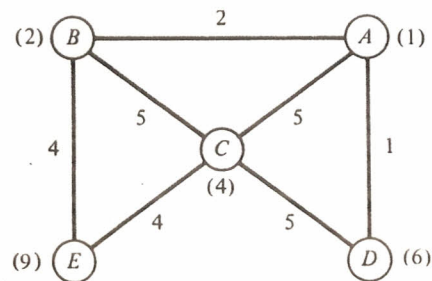
Once a call for service is received, and provided that an ambulance is available, an ambulance is immediately dispatched to the caller. The ambulance spends exactly 4 minutes at the scene of the call for *all calls* and then travels back to the medical facility (travel takes exactly the same time both ways). As soon as an ambulance returns to its origin, it immediately becomes available to respond to the next call.

Calls for service that are received when no ambulances are available are placed in a first-come, first-served queue and eventually receive service. No calls for service are ever lost.

a. Show that if it is desired to minimize the *average service time* on this network (where a service time consists of the round trip travel time *plus* the time spent on the scene) the two separate medical facilities should be placed at two nodes of the network.

b. The optimal locations of the two facilities have now been determined with the objective of part (a) in mind. Determine the minimum number of ambulances to be placed at each one of the two facilities if the whole system (i.e., the ambulance dispatching process from the two centers) is ever to *reach steady state*. (The minimum number need not be the same for both facilities.) Please explain your work and reasoning clearly.

c. How would your answer to part (b) change if the duration of time spent on the scene of each call for service were, instead, a random variable S, with pdf

$$f_S(s) = 15e^{-15s} \quad \text{for } s \geq 0$$

where S is given in hours? Please explain your answer.

6.14 *Review of several problems*   Figure P6.14 shows a transportation network with 10 demand points (towns, centers, etc.)

a. What is the location of an emergency facility that minimizes expected travel time?

b. A regional planning committee wishes to designate "emergency artery" roads so that *all* the demand points are connected in time of snow. If the criterion is to select the minimum total length (in terms of travel time) of emergency artery roads, what are the corresponding roads?

c. What is the median and the absolute center with respect to the emergency artery roads only?
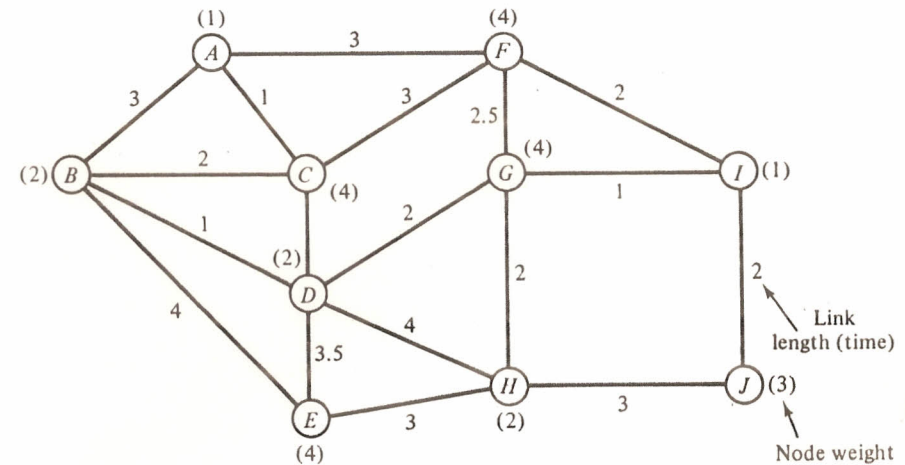
d. Note that nodes E, G, C, and F are demand points with large demand rate. By inspection, determine the minimum network (subgraph) that connects these points.

e. How will the solution in part (d) change if J is also included in the required subgraph?

f. Comment on the answers of parts (d) and (e). Is there any special network structure for these subgraphs?

g. Find a "good" traveling salesman tour on this graph. (Your tour need not be optimal and you may use any approach you wish.)

h. What is the length of the optimum Chinese postman tour of this network?

6.15 *Multiple delivery routes* The circulation manager of a small newspaper in a town is asked to reorganize the delivery of the newspaper. The travel network is shown in Figure P6.15.

The newspaper is produced at node 2 and must be delivered to all the other nodes. The manager should decide how many people must be hired to distribute the newspaper. In order to maximize service, the longest of all routes should be as short as possible, but in order to minimize costs the total length of the routes of all people delivering the newspapers should be as short as possible too. These two goals are, in general, contradictory! (Explain why.) The circulation manager meets the goals in the following way:
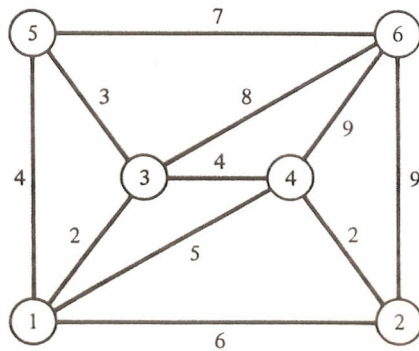
For all reasonable numbers of hired people he computes the routes such that the total length of the routes is minimized and each hired person has a route of non-zero length. He now compares the length of the longest route for the different solutions with different numbers of hired people. He then chooses the number of people N to hire, such that the longest route with N people is less than with $N - 1$ people, but is equal to the longest route with $N + 1$ people. What number is N?

6.16 *Design of an optimum road network considering congestion* It is well-known that travel time by car depends on the number of other cars on the same road. Keeping this in mind, let us consider the following simple problem.

We are given the network of Figure P6.16 where the travel time from node i to j, namely c_{ij}, depends on the flow (number of cars per unit time) x on that arc. Let:

$$c_{13} = 10x \qquad c_{32} = 50 + x \qquad c_{34} = 10 + x$$
$$c_{14} = 50 + x \qquad c_{42} = 10x$$

We now send a flow of 6 cars per unit time from node 1 to node 2.

a. As each driver tries to minimize his travel time independently from the other drivers, all possible routes (namely 1–3–2, 1–3–4–2 and 1–4–2) will have a flow density such that the route lengths (travel times) are equal. (Why?) Knowing this, how much flow will be on each arc and what is the travel time for all drivers?
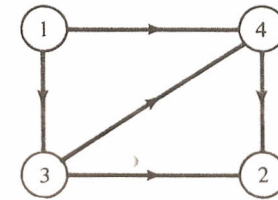
b. Assume now that someone, perhaps a police officer, is regulating the traffic. This means the police decide how much flow is allowed on each arc. Is it possible to regulate the traffic such that the travel time for *every* driver is lower than it was before, when each driver decided on his own? (The answer is *yes*!) How should the traffic be apportioned?

c. If the arc from node 3 to 4 did not exist and you were to decide if it should be built, what would you recommend?

6.17 *Facility location with queueing* Consider two small towns which are one unit distance apart, as shown in Figure P6.17. (Each town is represented as a single point (node) on this simple "network," i.e., intra-town distances can be considered insignificant).

A hospital equipped with a single ambulance is located at some point between the two towns which is a distance x away from the halfway point between the two towns.

Calls from the two towns that require dispatching of the ambulance occur in a Poisson manner at a *combined* rate $\lambda = 1/4$ calls/unit time. A fraction f_A of these calls come from Town A and a fraction f_B from Town B ($f_A + f_B = 1$).

In responding to each call the ambulance travels to the appropriate city at a constant speed v, spends a constant amount of time τ on the scene (picking up a patient) and returns to the hospital (with the patient) at the same constant speed v. Let $v = 1$ distance unit/time unit and $\tau = 1$ time unit.

Calls for ambulance dispatching queue up in a first-come, first-served manner until the ambulance eventually serves them. We define the "total response time" of the ambulance to a patient as the time interval between the instant when that patient calls for the ambulance and the instant when the patient arrives at the hospital.

a. Assuming steady-state conditions, find the expected total response time to a random patient. Your answer should be in terms of x, f_a and f_b only.

  Hint: To keep the algebra simple, write your answer in terms of x and of $(f_a - f_b)$.

Location of hospital
and ambulance
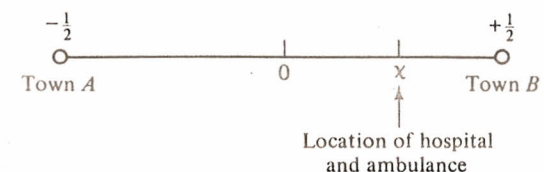
b. If the objective is to minimize the expected total response time per patient, what is the optimal value of x when $f_a = f_b = \frac{1}{2}$?

c. Does your answer in (b) agree with or violate Hakimi's theorem for the location of a median on networks? Please explain briefly.

d. In the general case (arbitrary λ, τ, v, f_a and f_b), does the question of whether steady-state is reached depend on the location of the hospital/ambulance? Please explain briefly (no mathematics).

e. Suppose now that all calls that find the ambulance busy, i.e. away from the hospital, are lost (e.g., the patients are transported to the hospital by taxi). Where should the hospital be located in this case if $f_a = 0.8$ and $f_b = 0.2$ (and, as before, $\lambda = \frac{1}{4}, \tau = 1, v = 1$) and the objective is still to minimize expected total response time *for those patients who are served by ambulance*? (Note that no queueing ever occurs in this case.)

f. Repeat part (e) assuming that an additional constraint is that no total response time should ever exceed 2.6 units (including the time on the scene).